Quantum groups in nature[1]

Jules Martel-Tordjman
(j.w. S. Bigelow)

Grenoble

---

[1] nature is actually homologies of labeled configuration spaces of surfaces

Quantum groups

$\Pi = \{\alpha_1, \ldots, \alpha_l\}$

The quantum algebra $U_q\mathfrak{g}$

- Let $\Pi := \{\alpha_1, \ldots, \alpha_l\}$ be a set of simple roots,

Quantum groups

$\Pi = \{\alpha_1, \dots, \alpha_l\}$

The quantum algebra $U_q\mathfrak{g}$

- Let $\Pi := \{\alpha_1, \dots, \alpha_l\}$ be a set of simple roots,
- and $C := (a_{i,j})_{1 \leqslant i,j \leqslant l}$ be a symmetrizable Cartan matrix
  ($a_{i,i} = 2,\ a_{i,j} \leqslant 0$ otherwise, $\exists (d_1, \dots, d_l) \in \mathbb{N}$ s.t. $(d_i a_{i,j})_{i,j}$ is symmetric)

The quantum algebra $U_q\mathfrak{g}$

- Let $\Pi := \{\alpha_1, \dots, \alpha_l\}$ be a set of simple roots,
- and $C := (a_{i,j})_{1 \leqslant i,j \leqslant l}$ be a symmetrizable Cartan matrix
  ($a_{i,i} = 2,\ a_{i,j} \leqslant 0$ otherwise, $\exists (d_1, \dots, d_l) \in \mathbb{N}$ s.t. $(d_i a_{i,j})_{i,j}$ is symmetric)
- Let $(\cdot, \cdot)$ be the sym. bilin. form (on $\mathbb{Z}\Pi$) given by the symmetric matrix.

Quantum groups

$\Pi = \{\alpha_1, \ldots, \alpha_l\}$

The quantum algebra $U_q\mathfrak{g}$

- Let $\Pi := \{\alpha_1, \ldots, \alpha_l\}$ be a set of simple roots,
- and $C := (a_{i,j})_{1 \leqslant i,j \leqslant l}$ be a symmetrizable Cartan matrix
  ($a_{i,i} = 2$, $a_{i,j} \leqslant 0$ otherwise, $\exists (d_1, \ldots, d_l) \in \mathbb{N}$ s.t. $(d_i a_{i,j})_{i,j}$ is symmetric)
- Let $(\cdot, \cdot)$ be the sym. bilin. form (on $\mathbb{Z}\Pi$) given by the symmetric matrix.

The quantum Kac–Moody algebra $U_q\mathfrak{g}$ associated with this data set is the $\mathbb{Q}(q)$-algebra generated by $E_\alpha, F_\alpha, K_\alpha^{\pm 1}$ (for all $\alpha \in \Pi$)

Quantum groups

$\Pi = \{\alpha_1, \ldots, \alpha_l\}$

$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$

The quantum algebra $U_q\mathfrak{g}$

- Let $\Pi := \{\alpha_1, \ldots, \alpha_l\}$ be a set of simple roots,
- and $C := (a_{i,j})_{1 \leqslant i,j \leqslant l}$ be a symmetrizable Cartan matrix
  ($a_{i,i} = 2$, $a_{i,j} \leqslant 0$ otherwise, $\exists (d_1, \ldots, d_l) \in \mathbb{N}$ s.t. $(d_i a_{i,j})_{i,j}$ is symmetric)
- Let $(\cdot, \cdot)$ be the sym. bilin. form (on $\mathbb{Z}\Pi$) given by the symmetric matrix.

The quantum Kac–Moody algebra $U_q\mathfrak{g}$ associated with this data set is the $\mathbb{Q}(q)$-algebra generated by $E_\alpha, F_\alpha, K_\alpha^{\pm 1}$ (for all $\alpha \in \Pi$), plus relations:

$$K_\alpha K_\alpha^{-1} = K_\alpha^{-1} K_\alpha = 1,$$

$$K_{\alpha_i} E_{\alpha_j} K_{\alpha_i}^{-1} = q_{\alpha_i}^{-\frac{a_{i,j}}{2}} E_{\alpha_j}, \qquad K_{\alpha_i} F_{\alpha_j}^{(1)} K_{\alpha_i}^{-1} = q_{\alpha_i}^{\frac{a_{i,j}}{2}} F_{\alpha_j}, \qquad [E_\alpha, F_\alpha] = \frac{K_\alpha - K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}}$$

where $q_\alpha := q^{d_\alpha}$, and the quantum Serre relations, for $\beta \neq \alpha$ and :

$$\sum_{l=0}^{1-a_{i,j}} (-1)^l \binom{1 - a_{i,j}}{l}_{q_\alpha} F_\alpha^l F_\beta F_\alpha^{1-a_{i,j}-l} = 0,$$

$$\sum_{l=0}^{1-a_{i,j}} (-1)^l \binom{1 - a_{i,j}}{l}_{q_\alpha} E_\alpha^l E_\beta E_\alpha^{1-a_{i,j}-l} = 0.$$

Quantum groups

$$\Pi = \{\alpha_1, \ldots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Quantum groups for topology

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi\rangle$$

Quantum groups for topology

$$\mathscr{B}_k = \left\langle \sigma_1, \dots, \sigma_k \middle| \begin{array}{c} \sigma_i\sigma_j = \sigma_j\sigma_i, |i-j| \geqslant 2, \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \end{array} \right\rangle \simeq \mathrm{Mod}(D_k)$$

Quantum groups

$$\Pi = \{\alpha_1, \ldots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Quantum groups for topology

$$\mathscr{B}_k = \left\langle \sigma_1, \ldots, \sigma_k \middle| \begin{array}{c} \sigma_i\sigma_j = \sigma_j\sigma_i, |i-j| \geqslant 2, \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \end{array} \right\rangle \simeq \mathrm{Mod}(D_k)$$

Input: $U_q\mathfrak{g}$-modules,

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Quantum groups for topology

$$\mathscr{B}_k = \left\langle \sigma_1, \dots, \sigma_k \,\middle|\, \begin{array}{c} \sigma_i\sigma_j = \sigma_j\sigma_i, |i-j| \geqslant 2, \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \end{array} \right\rangle \simeq \mathrm{Mod}(D_k)$$

Input: $U_q\mathfrak{g}$-modules,
Output: Reps. of $\mathscr{B}_k$ (using the R-matrix),

$$\mathscr{B}_k \qquad \mapsto \qquad \mathrm{End}_{U_q\mathfrak{sl}(2)}(V^{\otimes k}),$$

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Quantum groups for topology

$$\mathscr{B}_k = \left\langle \sigma_1, \dots, \sigma_k \middle| \begin{array}{c} \sigma_i\sigma_j = \sigma_j\sigma_i, |i-j| \geqslant 2, \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \end{array} \right\rangle \simeq \mathrm{Mod}(D_k)$$

Input: $U_q\mathfrak{g}$-modules,

Output: Reps. of $\mathscr{B}_k$ (using the R-matrix), invariants of links (using quantum/modified trace),

$$\begin{array}{ccc} \mathscr{B}_k & \mapsto & \mathrm{End}_{U_q\mathfrak{sl}(2)}(V^{\otimes k}), \\ \{ \text{ links } \} & \mapsto & \mathbb{C}[q^{\pm 1}], \end{array}$$

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Quantum groups for topology

$$\mathscr{B}_k = \left\langle \sigma_1, \dots, \sigma_k \middle| \begin{array}{l} \sigma_i\sigma_j = \sigma_j\sigma_i, |i-j| \geqslant 2, \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \end{array} \right\rangle \simeq \mathrm{Mod}(D_k)$$

Input: $U_q\mathfrak{g}$-modules, $q = \zeta = e^{2i\pi/r}$

Output: Reps. of $\mathscr{B}_k$ (using the R-matrix), invariants of links (using quantum/modified trace), of 3-manifolds (using Kirby colors) ...

$$\begin{array}{ccc} \mathscr{B}_k & \longmapsto & \mathrm{End}_{U_q\mathfrak{sl}(2)}(V^{\otimes k}), \\ \{\text{ links }\} & \longmapsto & \mathbb{C}[q^{\pm 1}], \\ \{\text{closed 3-manifolds}\} & \longmapsto & \mathbb{C}^* \end{array}$$

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Quantum groups for topology

$$\mathscr{B}_k = \left\langle \sigma_1, \dots, \sigma_k \middle| \begin{array}{l} \sigma_i\sigma_j = \sigma_j\sigma_i, |i-j| \geqslant 2, \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \end{array} \right\rangle \simeq \mathrm{Mod}(D_k)$$

Input: $U_q\mathfrak{g}$-modules, $q = \zeta = e^{2i\pi/r}$

Output: Reps. of $\mathscr{B}_k$ (using the R-matrix), invariants of links (using quantum/modified trace), of 3-manifolds (using Kirby colors) ...
TQFTs:

$$\mathrm{F}_\zeta : \begin{array}{ccc} \mathrm{3Cob} \text{ (cobordism cat.)} & \rightarrow & U_\zeta\mathfrak{sl}_2 - \text{mod.} \\ \mathscr{B}_k & \mapsto & \mathrm{End}_{U_q\mathfrak{sl}(2)}(V^{\otimes k}), \\ \{ \text{ links } \} & \mapsto & \mathbb{C}[q^{\pm 1}], \\ \{\text{closed 3-manifolds}\} & \mapsto & \mathbb{C}^* \end{array}$$

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Quantum groups for topology

$$\mathscr{B}_k = \left\langle \sigma_1, \dots, \sigma_k \middle| \begin{array}{c} \sigma_i\sigma_j = \sigma_j\sigma_i, |i-j| \geqslant 2, \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \end{array} \right\rangle \simeq \mathrm{Mod}(D_k)$$

Input: $U_q\mathfrak{g}$-modules, $q = \zeta = e^{2i\pi/r}$

Output: Reps. of $\mathscr{B}_k$ (using the R-matrix), invariants of links (using quantum/modified trace), of 3-manifolds (using Kirby colors) ...
TQFTs:

$$\mathrm{F}_\zeta : \begin{array}{ccc} 3\mathrm{Cob}\ \text{(cobordism cat.)} & \to & U_\zeta\mathfrak{sl}_2 - \text{mod.} \\ \mathscr{B}_k & \mapsto & \mathrm{End}_{U_q\mathfrak{sl}(2)}(V^{\otimes k}), \\ \{\ \text{links}\ \} & \mapsto & \mathbb{C}[q^{\pm 1}], \\ \{\text{closed 3-manifolds}\} & \mapsto & \mathbb{C}^* \\ S,\ \text{a surface} & \mapsto & V_S \end{array}$$

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi\rangle$$

Quantum groups for topology

$$\mathscr{B}_k = \left\langle \sigma_1, \dots, \sigma_k \,\middle|\, \begin{array}{c} \sigma_i\sigma_j = \sigma_j\sigma_i, |i-j| \geqslant 2, \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \end{array} \right\rangle \simeq \mathrm{Mod}(D_k)$$

Input: $U_q\mathfrak{g}$-modules, $q = \zeta = e^{2i\pi/r}$

Output: Reps. of $\mathscr{B}_k$ (using the R-matrix), invariants of links (using quantum/modified trace), of 3-manifolds (using Kirby colors) ... TQFTs:

$$\mathrm{F}_\zeta : \begin{array}{ccc} \text{3Cob (cobordism cat.)} & \to & U_\zeta\mathfrak{sl}_2 - \text{mod.} \\ \mathscr{B}_k & \mapsto & \mathrm{End}_{U_q\mathfrak{sl}(2)}(V^{\otimes k}), \\ \{\text{ links }\} & \mapsto & \mathbb{C}[q^{\pm 1}], \\ \{\text{closed 3-manifolds}\} & \mapsto & \mathbb{C}^* \\ S, \text{ a surface} & \mapsto & V_S \\ \mathrm{Mod}(S) & \mapsto & \mathrm{End}(V_S) \end{array}$$

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Quantum groups for topology

$$\mathscr{B}_k = \left\langle \sigma_1, \dots, \sigma_k \middle| \begin{array}{l} \sigma_i\sigma_j = \sigma_j\sigma_i, |i-j| \geqslant 2, \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \end{array} \right\rangle \simeq \mathrm{Mod}(D_k)$$

Input: $U_q\mathfrak{g}$-modules, $q = \zeta = e^{2i\pi/r}$

Output: Reps. of $\mathscr{B}_k$ (using the R-matrix), invariants of links (using quantum/modified trace), of 3-manifolds (using Kirby colors) ... TQFTs:

$$F_\zeta : \begin{array}{ccc} 3\mathrm{Cob} \text{ (cobordism cat.)} & \rightarrow & U_\zeta\mathfrak{sl}_2 - \text{mod.} \\ \mathscr{B}_k & \mapsto & \mathrm{End}_{U_q\mathfrak{sl}(2)}(V^{\otimes k}), \\ \{\text{ links }\} & \mapsto & \mathbb{C}[q^{\pm 1}], \\ \{\text{closed 3-manifolds}\} & \mapsto & \mathbb{C}^* \\ S, \text{ a surface} & \mapsto & V_S \\ \mathrm{Mod}(S) & \mapsto & \mathrm{End}(V_S) \end{array}$$

Question:

What topological content does $U_q\mathfrak{g}$ contain?

Quantum groups

$$\Pi = \{\alpha_1, \ldots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Quantum groups from topology

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Quantum groups from topology

Important history of *geometric models* for quantum groups, modules...: using Borel–Moore homologies of quiver varieties...
(Lusztig, Rouquier, Nakajima, Varagnolo–Vasserot, Maksimau–Stroppel...).

Quantum groups

$$\Pi = \{\alpha_1, \ldots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Quantum groups from topology

Important history of *geometric models* for quantum groups, modules...: using Borel–Moore homologies of quiver varieties... (Lusztig, Rouquier, Nakajima, Varagnolo–Vasserot, Maksimau–Stroppel...). They produce geometric categorifications.

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi\rangle$$

Quantum groups from topology

Important history of *geometric models* for quantum groups, modules...: using Borel–Moore homologies of quiver varieties...
(Lusztig, Rouquier, Nakajima, Varagnolo–Vasserot, Maksimau–Stroppel...).
They produce geometric categorifications.

Topological models:

Quantum groups

$$\Pi = \{\alpha_1, \ldots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Quantum groups from topology

Important history of *geometric models* for quantum groups, modules...: using Borel–Moore homologies of quiver varieties... (Lusztig, Rouquier, Nakajima, Varagnolo–Vasserot, Maksimau–Stroppel...). They produce geometric categorifications.

Topological models:

- (Bigelow) Recover modules on Iwahori–Hecke algebras from twisted homologies of conf spaces of disks. (following Lawrence)

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Quantum groups from topology

Important history of *geometric models* for quantum groups, modules...: using Borel–Moore homologies of quiver varieties... (Lusztig, Rouquier, Nakajima, Varagnolo–Vasserot, Maksimau–Stroppel...). They produce geometric categorifications.

Topological models:

- (Bigelow) Recover modules on Iwahori–Hecke algebras from twisted homologies of conf spaces of disks. (following Lawrence)
- (M.) Let V be the Verma module of $U_q\mathfrak{sl}(2)$.

$$\bigoplus_{n \in \mathbb{N}} \mathrm{H}_n^{\mathrm{BM}}(\mathrm{Conf}_n(D_k), S_n; \mathscr{R}) \simeq V^{\otimes k}$$

[will also be hidden in Anne-Laure's talk]

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi\rangle$$

Quantum groups from topology

Important history of *geometric models* for quantum groups, modules...: using Borel–Moore homologies of quiver varieties...

(Lusztig, Rouquier, Nakajima, Varagnolo–Vasserot, Maksimau–Stroppel...).

They produce geometric categorifications.

Topological models:

- (Bigelow) Recover modules on Iwahori–Hecke algebras from twisted homologies of conf spaces of disks. (following Lawrence)
- (M.) Let V be the Verma module of $U_q\mathfrak{sl}(2)$.

$$\bigoplus_{n\in\mathbb{N}} \mathrm{H}_n^{\mathrm{BM}}(\mathrm{Conf}_n(D_k), S_n; \mathscr{R}) \simeq V^{\otimes k}$$

[will also be hidden in Anne-Laure's talk]

- (De Renzi–M.) Let ad be the adjoint rep of $u_\zeta\mathfrak{sl}(2)$, Σ_g^1 the genus g surface

$$\bigoplus_{n} \mathrm{H}_n^{\mathrm{BM}}(\mathrm{Conf}_n(\Sigma_g^1), S_n; V_n) \simeq \mathrm{ad}^{\otimes g}$$

Quantum groups

$\Pi = \{\alpha_1, \ldots, \alpha_l\}$

$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$

Quantum groups from topology

Important history of *geometric models* for quantum groups, modules...: using Borel–Moore homologies of quiver varieties... (Lusztig, Rouquier, Nakajima, Varagnolo–Vasserot, Maksimau–Stroppel...). They produce geometric categorifications.

Topological models:

- (Bigelow) Recover modules on Iwahori–Hecke algebras from twisted homologies of conf spaces of disks. (following Lawrence)
- (M.) Let V be the Verma module of $U_q\mathfrak{sl}(2)$.

$$\bigoplus_{n\in\mathbb{N}} \mathrm{H}_n^{\mathrm{BM}}(\mathrm{Conf}_n(D_k), S_n; \mathscr{R}) \simeq V^{\otimes k}$$

  [will also be hidden in Anne-Laure's talk]
- (De Renzi–M.) Let ad be the adjoint rep of $u_\zeta\mathfrak{sl}(2)$, Σ_g^1 the genus g surface

$$\bigoplus_{n} \mathrm{H}_n^{\mathrm{BM}}(\mathrm{Conf}_n(\Sigma_g^1), S_n; V_n) \simeq \mathrm{ad}^{\otimes g}$$

- (Godfard) Products of simple $U_\zeta\mathfrak{sl}(2)$ modules in $\bigoplus_{n\in\mathbb{N}} \mathrm{H}_n(\mathrm{Conf}_n(D_k))$

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Quantum groups from topology

Important history of *geometric models* for quantum groups, modules...: using Borel–Moore homologies of quiver varieties... (Lusztig, Rouquier, Nakajima, Varagnolo–Vasserot, Maksimau–Stroppel...). They produce geometric categorifications.

Topological models:

- (Bigelow) Recover modules on Iwahori–Hecke algebras from twisted homologies of conf spaces of disks. (following Lawrence)
- (M.) Let V be the Verma module of $U_q\mathfrak{sl}(2)$.

$$\bigoplus_{n\in\mathbb{N}} \mathrm{H}_n^{\mathrm{BM}}(\mathrm{Conf}_n(D_k), S_n; \mathscr{R}) \simeq V^{\otimes k}$$

[will also be hidden in Anne-Laure's talk]

- (De Renzi–M.) Let ad be the adjoint rep of $u_\zeta\mathfrak{sl}(2)$, Σ_g^1 the genus g surface

$$\bigoplus_{n} \mathrm{H}_n^{\mathrm{BM}}(\mathrm{Conf}_n(\Sigma_g^1), S_n; V_n) \simeq \mathrm{ad}^{\otimes g}$$

- (Godfard) Products of simple $U_\zeta\mathfrak{sl}(2)$ modules in $\bigoplus_{n\in\mathbb{N}} \mathrm{H}_n(\mathrm{Conf}_n(D_k))$

Everywhere a mapping class group TQFT representation is intertwined

$\Pi = \{\alpha_1, \dots, \alpha_l\}$

$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$

The quantum algebra $U_q\mathfrak{g}$

- Let $\Pi := \{\alpha_1, \dots, \alpha_l\}$ be a set of simple roots,
- and $C := (a_{i,j})_{1 \leqslant i,j \leqslant l}$ be a symmetrizable Cartan matrix
  $(a_{i,i} = 2,\ a_{i,j} \leqslant 0$ otherwise, $\exists (d_1, \dots, d_l) \in \mathbb{N}$ s.t. $(d_i a_{i,j})_{i,j}$ is symmetric)
- Let $(\cdot, \cdot)$ be the sym. bilin. form (on $\mathbb{Z}\Pi$) given by the symmetric matrix.

The quantum Kac–Moody algebra $U_q\mathfrak{g}$ associated with this data set is the $\mathbb{Q}(q)$-algebra generated by $E_\alpha, F_\alpha, K_\alpha^{\pm 1}$ (for all $\alpha \in \Pi$) , plus relations:

$$K_\alpha K_\alpha^{-1} = K_\alpha^{-1} K_\alpha = 1,$$

$$K_{\alpha_i} E_{\alpha_j} K_{\alpha_i}^{-1} = q_{\alpha_i}^{-\frac{a_{i,j}}{2}} E_{\alpha_j}, \qquad K_{\alpha_i} F_{\alpha_j}^{(1)} K_{\alpha_i}^{-1} = q_{\alpha_i}^{\frac{a_{i,j}}{2}} F_{\alpha_j}, \qquad [E_\alpha, F_\alpha] = \frac{K_\alpha - K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}}$$

where $q_\alpha := q^{d_\alpha}$, and the quantum Serre relations, for $\beta \neq \alpha$ and :

$$\sum_{l=0}^{1-a_{i,j}} (-1)^l \binom{1 - a_{i,j}}{l}_{q_\alpha} F_\alpha^l F_\beta F_\alpha^{1-a_{i,j}-l} = 0,$$

$$\sum_{l=0}^{1-a_{i,j}} (-1)^l \binom{1 - a_{i,j}}{l}_{q_\alpha} E_\alpha^l E_\beta E_\alpha^{1-a_{i,j}-l} = 0.$$

Quantum groups

$$\Pi = \{\alpha_1, \ldots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

The quantum algebra $U_q\mathfrak{g}^{<0}$

- Let $\Pi := \{\alpha_1, \ldots, \alpha_l\}$ be a set of simple roots,
- and $C := (a_{i,j})_{1 \leqslant i,j \leqslant l}$ be a symmetrizable Cartan matrix ($a_{i,i} = 2,\ a_{i,j} \leqslant 0$ otherwise, $\exists (d_1, \ldots, d_l) \in \mathbb{N}$ s.t. $(d_i a_{i,j})_{i,j}$ is symmetric)
- Let $(\cdot, \cdot)$ be the sym. bilin. form (on $\mathbb{Z}\Pi$) given by the symmetric matrix.

The quantum Kac–Moody algebra $U_q\mathfrak{g}^{<0}$ associated with this data set is the $\mathbb{Q}(q)$-algebra generated by $F_\alpha, E_\alpha, K_\alpha^{\pm 1}$ for all $\alpha \in \Pi$ with the relations:

$$K_\alpha K_\alpha^{-1} = K_\alpha^{-1} K_\alpha = 1,$$

$$K_{\alpha_i} E_{\alpha_j} K_{\alpha_i}^{-1} = q_{\alpha_i}^{-\frac{a_{i,j}}{2}} E_{\alpha_j}, \qquad K_{\alpha_i} F_{\alpha_j}^{(1)} K_{\alpha_i}^{-1} = q_{\alpha_i}^{\frac{a_{i,j}}{2}} F_{\alpha_j}, \qquad [E_\alpha, F_\alpha] = \frac{K_\alpha - K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}},$$

where $q_\alpha := q^{d_\alpha}$, and the quantum Serre relations, for $\beta \neq \alpha$ and :

$$\sum_{l=0}^{1-a_{i,j}} (-1)^l \binom{1-a_{i,j}}{l}_{q_\alpha} F_\alpha^l F_\beta F_\alpha^{1-a_{i,j}-l} = 0,$$

$$\sum_{l=0}^{1-a_{i,j}} (-1)^l \binom{1-a_{i,j}}{l}_{q_\alpha} E_\alpha^l E_\beta E_\alpha^{1-a_{i,j}-l} = 0.$$

Quantum groups

$$\Pi = \{\alpha_1, \ldots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Configurations of points decorated by simple roots

Quantum groups

$$\Pi = \{\alpha_1, \ldots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Configurations of points decorated by simple roots

- Recall $\Pi = \{\alpha_1, \ldots, \alpha_l\}$, the set of simple roots.

Quantum groups

$\Pi = \{\alpha_1, \ldots, \alpha_l\}$

$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$

Configurations of points decorated by simple roots

- Recall $\Pi = \{\alpha_1, \ldots, \alpha_l\}$, the set of simple roots.
- Let $\{c : \Pi \to \mathbb{N}\} \simeq \{\sum_i n_i\alpha_i, n_i \in \mathbb{N}\} =: \mathbb{N}\Pi$.

Quantum groups

$$\Pi = \{\alpha_1, \ldots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$D \quad =$ 

Configurations of points decorated by simple roots

- Recall $\Pi = \{\alpha_1, \ldots, \alpha_l\}$, the set of simple roots.
- Let $\{c : \Pi \to \mathbb{N}\} \simeq \{\sum_i n_i \alpha_i, n_i \in \mathbb{N}\} =: \mathbb{N}\Pi$.

$D :=$ 

Quantum groups

$$\Pi = \{\alpha_1, \ldots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi\rangle$$

Homology

$D \quad =$

Configurations of points decorated by simple roots

- Recall $\Pi = \{\alpha_1, \ldots, \alpha_l\}$, the set of simple roots.
- Let $\{c : \Pi \to \mathbb{N}\} \simeq \{\sum_i n_i\alpha_i, n_i \in \mathbb{N}\} =: \mathbb{N}\Pi$.

$D :=$

For $c \in \mathbb{N}\Pi$, the <u>c-colored configuration space of D</u> is defined as follows:

$$\mathrm{Conf}_c(D) := \left(D^{|c|} \setminus \bigcup_{i<j} \{z_i = z_j\} \right) \Big/ \mathfrak{S}_{c(\alpha_1)} \times \cdots \times \mathfrak{S}_{c(\alpha_l)}$$

where D is a closed disk and $|\sum_i n_i\alpha_i| := \sum_i n_i$.

Quantum groups

$$\Pi = \{\alpha_1, \ldots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi\rangle$$

Homology

$D =$ 

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \ldots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \ldots, z_{c(\alpha_i)}^{\alpha_i}\}$$

Configurations of points decorated by simple roots

- Recall $\Pi = \{\alpha_1, \ldots, \alpha_l\}$, the set of simple roots.
- Let $\{c : \Pi \to \mathbb{N}\} \simeq \{\sum_i n_i\alpha_i, n_i \in \mathbb{N}\} =: \mathbb{N}\Pi$.

$D :=$ 

For $c \in \mathbb{N}\Pi$, the c-colored configuration space of D is defined as follows:

$$\mathrm{Conf}_c(D) := \left(D^{|c|} \setminus \bigcup_{i<j} \{z_i = z_j\} \right) \Big/ \mathfrak{S}_{c(\alpha_1)} \times \cdots \times \mathfrak{S}_{c(\alpha_l)}$$

where D is a closed disk and $|\sum_i n_i\alpha_i| := \sum_i n_i$.

An element of Conf_c is denoted

$$\left(\{z_1^{\alpha_1}, \ldots, z_{c(\alpha_1)}^{\alpha_1}\}, \{z_1^{\alpha_2}, \ldots, z_{c(\alpha_2)}^{\alpha_2}\}, \ldots, \{z_1^{\alpha_l}, \ldots, z_{c(\alpha_l)}^{\alpha_l}\}\right)$$

and can be seen as l packages of indistinguishable points

Quantum groups

$$\Pi = \{\alpha_1, \ldots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z^{\alpha_1}}, \ldots, \boldsymbol{z^{\alpha_l}})\}$$

$$\boldsymbol{z^{\alpha_i}} = \{z_1^{\alpha_i}, \ldots, z_{c(\alpha_i)}^{\alpha_i}\}$$

Computing angles

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D \quad =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

Computing angles

Let $\{z_1, \dots z_n\}, \{w_1, \dots, w_k\}$ be two configurations of D and $\mathbb{S}^1$ the unit circle.

$$f\left(\{z_1, \dots, z_n\}\right) := \prod_{1<i<j<n} \left(\frac{z_j - z_i}{|z_j - z_i|} \right)^2 = \prod_{1<i,j<n} \left(\frac{z_j - z_i}{|z_j - z_i|} \right) \in \mathbb{S}^1, \qquad (1)$$

$$g\left(\{z_1, \dots, z_n\}, \{w_1, \dots, w_k\}\right) := \prod_{1<i<n, 1<j<k} \left(\frac{w_j - z_i}{|w_j - z_i|} \right)^2 \in \mathbb{S}^1, \qquad (2)$$

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$D =$ 

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

Computing angles

Let $\{z_1, \dots z_n\}, \{w_1, \dots, w_k\}$ be two configurations of D and $\mathbb{S}^1$ the unit circle.

$$f(\{z_1, \dots, z_n\}) := \prod_{1<i<j<n} \left(\frac{z_j - z_i}{|z_j - z_i|} \right)^2 = \prod_{1<i,j<n} \left(\frac{z_j - z_i}{|z_j - z_i|} \right) \in \mathbb{S}^1, \quad (1)$$

$$g(\{z_1, \dots, z_n\}, \{w_1, \dots, w_k\}) := \prod_{1<i<n, 1<j<k} \left(\frac{w_j - z_i}{|w_j - z_i|} \right)^2 \in \mathbb{S}^1, \quad (2)$$

Let $c \in \mathbb{N}\Pi$:

$$\Phi_c : \begin{array}{ccc} \mathrm{Conf}_c & \to & (\mathbb{S}^1)^l \times (\mathbb{S}^1)^{\frac{l(l-1)}{2}} =: B_c \\ (\boldsymbol{z}^{\alpha_1}, \dots, \boldsymbol{z}^{\alpha_l}) & \mapsto & (f(\boldsymbol{z}^{\alpha_1}), \dots, f(\boldsymbol{z}^{\alpha_l}), g(\boldsymbol{z}^{\alpha_1}, \boldsymbol{z}^{\alpha_2}), \dots, g(\boldsymbol{z}^{\alpha_{l-1}}, \boldsymbol{z}^{\alpha_l})) \end{array}$$

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$D =$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

Computing angles

Let $\{z_1, \dots z_n\}, \{w_1, \dots, w_k\}$ be two configurations of D and $\mathbb{S}^1$ the unit circle.

$$f\left(\{z_1, \dots, z_n\}\right) := \prod_{1<i<j<n} \left(\frac{z_j - z_i}{|z_j - z_i|} \right)^2 = \prod_{1<i,j<n} \left(\frac{z_j - z_i}{|z_j - z_i|} \right) \in \mathbb{S}^1, \quad (1)$$

$$g\left(\{z_1, \dots, z_n\}, \{w_1, \dots, w_k\}\right) := \prod_{1<i<n, 1<j<k} \left(\frac{w_j - z_i}{|w_j - z_i|} \right)^2 \in \mathbb{S}^1, \quad (2)$$

Let $c \in \mathbb{N}\Pi$:

$$\Phi_c : \begin{array}{ccc} \mathrm{Conf}_c & \to & (\mathbb{S}^1)^l \times (\mathbb{S}^1)^{\frac{l(l-1)}{2}} =: B_c \\ (\boldsymbol{z}^{\alpha_1}, \dots, \boldsymbol{z}^{\alpha_l}) & \mapsto & (f(\boldsymbol{z}^{\alpha_1}), \dots, f(\boldsymbol{z}^{\alpha_l}), g(\boldsymbol{z}^{\alpha_1}, \boldsymbol{z}^{\alpha_2}), \dots, g(\boldsymbol{z}^{\alpha_{l-1}}, \boldsymbol{z}^{\alpha_l})) \end{array}$$

Let $\pi_c := \pi_1(B_c, (1, \dots, 1)) = \mathbb{Z}^l \times \mathbb{Z}^{\frac{l(l-1)}{2}}$ for which we choose generators:

$$(k_1, \dots, k_l, k_{(1,2)}, \dots, k_{(l-1,l)})$$

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$D =$ 

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

Computing angles

Let $\{z_1, \dots z_n\}, \{w_1, \dots, w_k\}$ be two configurations of D and $\mathbb{S}^1$ the unit circle.

$$f(\{z_1, \dots, z_n\}) := \prod_{1<i<j<n} \left(\frac{z_j - z_i}{|z_j - z_i|} \right)^2 = \prod_{1<i,j<n} \left(\frac{z_j - z_i}{|z_j - z_i|} \right) \in \mathbb{S}^1, \qquad (1)$$

$$g(\{z_1, \dots, z_n\}, \{w_1, \dots, w_k\}) := \prod_{1<i<n, 1<j<k} \left(\frac{w_j - z_i}{|w_j - z_i|} \right)^2 \in \mathbb{S}^1, \qquad (2)$$

Let $c \in \mathbb{N}\Pi$:

$$\Phi_c : \begin{array}{ccc} \mathrm{Conf}_c & \to & (\mathbb{S}^1)^l \times (\mathbb{S}^1)^{\frac{l(l-1)}{2}} =: B_c \\ (\boldsymbol{z}^{\alpha_1}, \dots, \boldsymbol{z}^{\alpha_l}) & \mapsto & (f(\boldsymbol{z}^{\alpha_1}), \dots, f(\boldsymbol{z}^{\alpha_l}), g(\boldsymbol{z}^{\alpha_1}, \boldsymbol{z}^{\alpha_2}), \dots, g(\boldsymbol{z}^{\alpha_{l-1}}, \boldsymbol{z}^{\alpha_l})) \end{array}$$

Let $\pi_c := \pi_1(B_c, (1, \dots, 1)) = \mathbb{Z}^l \times \mathbb{Z}^{\frac{l(l-1)}{2}}$ for which we choose generators:

$$(k_1, \dots, k_l, k_{(1,2)}, \dots, k_{(l-1,l)})$$

Coordinates of Φ_c pushed to π_1's computes winding numbers of loops

Quantum groups

$$\Pi = \{\alpha_1, \ldots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D \quad =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z^{\alpha_1}}, \ldots, \boldsymbol{z^{\alpha_l}})\}$$

$$\boldsymbol{z^{\alpha_i}} = \{z_1^{\alpha_i}, \ldots, z_{c(\alpha_i)}^{\alpha_i}\}$$

A local system: monodromy encoding the Cartan matrix

$$\Phi_c : \quad \mathrm{Conf}_c \quad \to \quad (\mathbb{S}^1)^l \times (\mathbb{S}^1)^{\frac{l(l-1)}{2}} =: B_c$$

and $\pi_c := \pi_1(B_c, (1, \ldots, 1)) = \mathbb{Z}^l \times \mathbb{Z}^{\frac{l(l-1)}{2}} = \langle k_1, \ldots, k_l, k_{(1,2)}, \ldots, k_{(l-1,l)} \rangle$.

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

A local system: monodromy encoding the Cartan matrix

$$\Phi_c : \quad \mathrm{Conf}_c \quad \rightarrow \quad (\mathbb{S}^1)^l \times (\mathbb{S}^1)^{\frac{l(l-1)}{2}} =: B_c$$

and $\pi_c := \pi_1(B_c, (1, \dots, 1)) = \mathbb{Z}^l \times \mathbb{Z}^{\frac{l(l-1)}{2}} = \langle k_1, \dots, k_l, k_{(1,2)}, \dots, k_{(l-1,l)} \rangle$.

Definition

$$\rho_c : \begin{cases} \pi_c & \rightarrow & \mathscr{R} := \mathbb{Z}\left[q^{\pm 1}\right] \\ k_i & \mapsto & -q^{(\alpha_i,\alpha_i)/2} = -q^{d_i} = -q_{\alpha_i} \\ k_{(i,j)} & \mapsto & q^{(\alpha_i,\alpha_j)/2} =: q_{\alpha_i,\alpha_j}. \end{cases}$$

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D \; =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

A local system: monodromy encoding the Cartan matrix

$$\Phi_c : \quad \mathrm{Conf}_c \quad \to \quad (\mathbb{S}^1)^l \times (\mathbb{S}^1)^{\frac{l(l-1)}{2}} =: B_c$$

and $\pi_c := \pi_1(B_c, (1, \dots, 1)) = \mathbb{Z}^l \times \mathbb{Z}^{\frac{l(l-1)}{2}} = \langle k_1, \dots, k_l, k_{(1,2)}, \dots, k_{(l-1,l)} \rangle$.

Definition

$$\rho_c : \left\{ \begin{array}{rcl} \pi_c & \to & \mathscr{R} := \mathbb{Z}\left[q^{\pm 1}\right] \\ k_i & \mapsto & -q^{(\alpha_i, \alpha_i)/2} = -q^{d_i} = -q_{\alpha_i} \\ k_{(i,j)} & \mapsto & q^{(\alpha_i, \alpha_j)/2} =: q_{\alpha_i, \alpha_j}. \end{array} \right.$$

Then the set $\mathscr{R}_c := (\mathscr{R}, \mathscr{W}_c = \Phi_c \circ r_{\frac{\pi}{2}}, \rho_c)$ endows Conf_c with a local system with fiber isom. to $\mathscr{R}$ and monodromy given by $\rho_c \circ \mathscr{W}_c$.

Quantum groups

$$\Pi = \{\alpha_1, \ldots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D \quad =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \ldots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$
$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \ldots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

A local system: monodromy encoding the Cartan matrix

$$\Phi_c : \quad \mathrm{Conf}_c \quad \to \quad (\mathbb{S}^1)^l \times (\mathbb{S}^1)^{\frac{l(l-1)}{2}} =: B_c$$

and $\pi_c := \pi_1(B_c, (1, \ldots, 1)) = \mathbb{Z}^l \times \mathbb{Z}^{\frac{l(l-1)}{2}} = \langle k_1, \ldots, k_l, k_{(1,2)}, \ldots, k_{(l-1,l)} \rangle$.

Definition

$$\rho_c : \begin{cases} \pi_c & \to & \mathscr{R} := \mathbb{Z}\left[q^{\pm 1}\right] \\ k_i & \mapsto & -q^{(\alpha_i,\alpha_i)/2} = -q^{d_i} = -q_{\alpha_i} \\ k_{(i,j)} & \mapsto & q^{(\alpha_i,\alpha_j)/2} =: q_{\alpha_i,\alpha_j}. \end{cases}$$

Then the set $\mathscr{R}_c := (\mathscr{R}, \mathscr{W}_c = \Phi_c \circ r_{\frac{\pi}{2}}, \rho_c)$ endows Conf_c with a local system with fiber isom. to $\mathscr{R}$ and monodromy given by $\rho_c \circ \mathscr{W}_c$.

- A path from/to a vertically aligned configuration where two particles decorated by α_i wind once around each other is sent to $-q^{d_i}$,
- A path from/to a vertically aligned configuration where two particles decorated resp. by α_i, α_j wind once around each other is sent to $q^{(\alpha_i,\alpha_j)/2}$.

Quantum groups

$$\Pi = \{\alpha_1, \ldots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$D =$

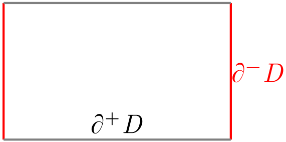
$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \ldots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \ldots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

Twisted homologies

$D =$



with $\partial^- D \subset \partial D$ in red,

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$D =$

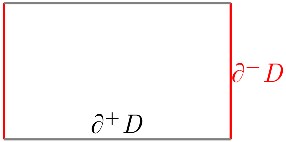
$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

Twisted homologies

$D =$



with $\partial^- D \subset \partial D$ in red, Let $c \in \mathbb{N}\Pi$,

$S_c^- \subset \partial\,\mathrm{Conf}_c(D)$ is the set of configurations w. at least one point in $\partial^- D$.

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi\rangle$$

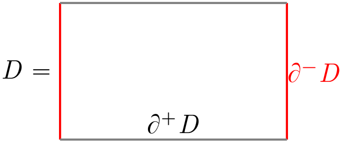
Homology

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

Twisted homologies



with $\partial^- D \subset \partial D$ in red, Let $c \in \mathbb{N}\Pi$,

$S_c^- \subset \partial\,\mathrm{Conf}_c(D)$ is the set of configurations w. at least one point in $\partial^- D$.

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c^-; \mathscr{R}_c),$$

$$\mathscr{H}_c := \mathrm{H}_{|c|}(\mathrm{Conf}_c, S_c^-; \mathscr{R}_c),$$

where BM indicates Borel–Moore homology

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$D =$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

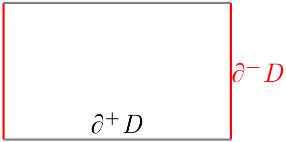
$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

Twisted homologies

$D =$



with $\partial^- D \subset \partial D$ in red,   Let $c \in \mathbb{N}\Pi$,

$S_c^- \subset \partial\,\mathrm{Conf}_c(D)$ is the set of configurations w. at least one point in $\partial^- D$.

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c^-; \mathscr{R}_c),$$
$$\mathscr{H}_c := \mathrm{H}_{|c|}(\mathrm{Conf}_c, S_c^-; \mathscr{R}_c),$$

where BM indicates Borel–Moore homology, both carrying an $\mathscr{R}$-module structure.

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D \quad =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

Example of Borel–Moore classes

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D \quad =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

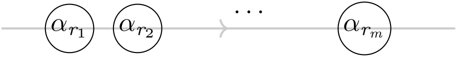
$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

Example of Borel–Moore classes

- A pearl necklace:



$$\in \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c(I))$$

where I is the (open) unit interval and $c = \alpha_{r_1} + \cdots + \alpha_{r_m} \in \mathbb{N}\Pi$. Defined from an embedding of the open simplex

$$\Delta^{|c|} := \{0 < t_1 < \cdots < t_{|c|} < 1\} = \mathrm{OConf}_c(I)$$

.

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$D \; =$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

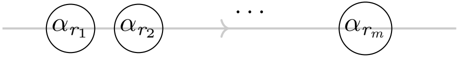
$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

Example of Borel–Moore classes

- A pearl necklace:



$$\in \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c(I))$$

where I is the (open) unit interval and $c = \alpha_{r_1} + \cdots + \alpha_{r_m} \in \mathbb{N}\Pi$. Defined from an embedding of the open simplex

$$\Delta^{|c|} := \{0 < t_1 < \cdots < t_{|c|} < 1\} = \mathrm{OConf}_c(I)$$

.

- (a pearl necklace) $\hookrightarrow (D, \partial^- D)$ defines $\mathrm{Conf}_c(I) \hookrightarrow (\mathrm{Conf}_c(D), S_c^-)$.

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D \; =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

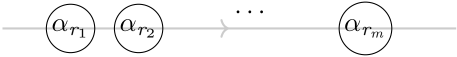
$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

Example of Borel–Moore classes

- A pearl necklace:



$$\in \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c(I))$$

where I is the (open) unit interval and $c = \alpha_{r_1} + \dots + \alpha_{r_m} \in \mathbb{N}\Pi$. Defined from an embedding of the open simplex

$$\Delta^{|c|} := \{0 < t_1 < \dots < t_{|c|} < 1\} = \mathrm{OConf}_c(I)$$

.

- (a pearl necklace) $\hookrightarrow (D, \partial^- D)$ defines $\mathrm{Conf}_c(I) \hookrightarrow (\mathrm{Conf}_c(D), S_c^-)$. The corresp. diagram represents the image of the pearl necklace by

$$\mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c(I)) \to \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c(D), S_c^-)$$

.

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$D =$ 

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

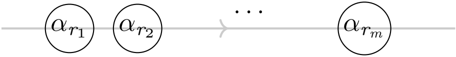
$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

Example of Borel–Moore classes

- A pearl necklace:



$$\in \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c(I))$$

where I is the (open) unit interval and $c = \alpha_{r_1} + \cdots + \alpha_{r_m} \in \mathbb{N}\Pi$. Defined from an embedding of the open simplex

$$\Delta^{|c|} := \{0 < t_1 < \cdots < t_{|c|} < 1\} = \mathrm{OConf}_c(I)$$

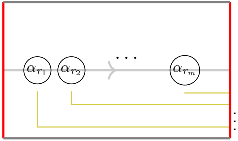
.

- (a pearl necklace) $\hookrightarrow (D, \partial^- D)$ defines $\mathrm{Conf}_c(I) \hookrightarrow (\mathrm{Conf}_c(D), S_c^-)$. The corresp. diagram represents the image of the pearl necklace by

$$\mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c(I)) \to \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c(D), S_c^-)$$

.

- $\mathscr{F}_{(r_1, \dots, r_m)} :=$



$$\in \mathscr{H}_c^{\mathrm{BM}}$$

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D \ = $$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

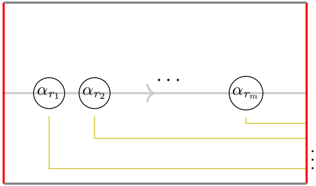
$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

Structure of the Borel–Moore homologies

$$\mathscr{F}_{(r_1,\dots,r_m)} :=$$



$$\in \mathscr{H}_c^{\mathrm{BM}}$$

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$D \quad =$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

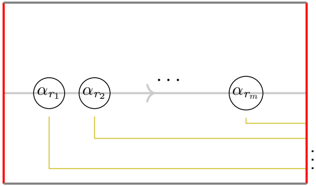
$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

Structure of the Borel–Moore homologies

$\mathscr{F}_{(r_1,\dots,r_m)} :=$



$\in \mathscr{H}_c^{\mathrm{BM}}$

Proposition (Bigelow–M.)

Let $c \in \mathbb{N}\Pi$, then the module $\mathscr{H}_c^{\mathrm{BM}}$ is:

- *a free $\mathscr{R}$-module,*

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$D =$

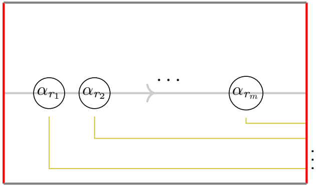

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

Structure of the Borel–Moore homologies

$\mathscr{F}_{(r_1,\dots,r_m)} :=$


$\in \mathscr{H}_c^{\mathrm{BM}}$

Proposition (Bigelow–M.)

Let $c \in \mathbb{N}\Pi$, then the module $\mathscr{H}_c^{\mathrm{BM}}$ is:

- *a free $\mathscr{R}$-module,*
- *for which the set $\mathscr{B}_{\mathscr{H}_c^{\mathrm{BM}}} := \left\{\mathscr{F}_{(r_1,\dots,r_m)} \text{ s.t. } \sum \alpha_{r_i} = c\right\}$ is a basis.*

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D \quad =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

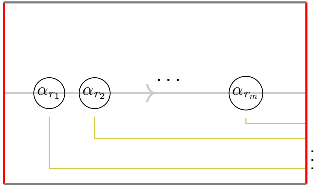
$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

Structure of the Borel–Moore homologies

$$\mathscr{F}_{(r_1,\dots,r_m)} :=$$



$$\in \mathscr{H}_c^{\mathrm{BM}}$$

Proposition (Bigelow–M.)

Let $c \in \mathbb{N}\Pi$, then the module $\mathscr{H}_c^{\mathrm{BM}}$ is:

- *a free $\mathscr{R}$-module,*
- *for which the set $\mathscr{B}_{\mathscr{H}_c^{\mathrm{BM}}} := \left\{\mathscr{F}_{(r_1,\dots,r_m)} \text{ s.t. } \sum \alpha_{r_i} = c\right\}$ is a basis.*
- *It is the only non-vanishing module from the sequence $\mathrm{H}_\bullet^{\mathrm{BM}}(\mathrm{Conf}_c, S_c^-; \mathscr{R}_c)$. (the homology is concentrated in the middle dimension).*

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D \quad =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$
$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D \quad =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

A product

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D \quad =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

A product

Stacking disks from above:

Quantum groups

$$\Pi = \{\alpha_1, \ldots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D \quad =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \ldots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \ldots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

A product

Stacking disks from above:

provides a map: $\mathrm{Conf}_{c_1} \times \mathrm{Conf}_{c_2} \to \mathrm{Conf}_{c_1+c_2}$,

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D \quad =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

A product

Stacking disks from above:

provides a map: $\mathrm{Conf}_{c_1} \times \mathrm{Conf}_{c_2} \to \mathrm{Conf}_{c_1+c_2}$, which gives at homology :

$$\mathscr{H}^{\epsilon}_{c_1} \otimes \mathscr{H}^{\epsilon}_{c_2} \to \mathscr{H}^{\epsilon}_{c_1+c_2}.$$

(ϵ indicates Borel–Moore or not)

Quantum groups

$$\Pi = \{\alpha_1, \ldots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$D =$ 

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \ldots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \ldots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

A product

Stacking disks from above:

provides a map: $\mathrm{Conf}_{c_1} \times \mathrm{Conf}_{c_2} \to \mathrm{Conf}_{c_1+c_2}$, which gives at homology :

$$\mathscr{H}_{c_1}^{\epsilon} \otimes \mathscr{H}_{c_2}^{\epsilon} \to \mathscr{H}_{c_1+c_2}^{\epsilon}.$$

(ϵ indicates Borel–Moore or not)

Proposition

The space:

$$\mathscr{H}^{\epsilon} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c^{\epsilon}$$

is an algebra (graded by the monoid $\mathbb{N}\Pi$*).*

Quantum groups

$$\Pi = \{\alpha_1, \ldots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi\rangle$$

Homology

$$D \quad =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \ldots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \ldots, z_{c(\alpha_i)}^{\alpha_i}\}$$

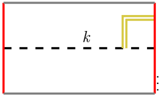
$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

Didived powers at Borel–Moore homology

$$\mathscr{F}_\alpha^{(k)} := q^{-d_\alpha \frac{k(k-1)}{4}}$$



$$\in \mathscr{H}_{k\alpha}^{\mathrm{BM}}$$

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi\rangle$$

Homology

$$D =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

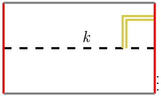
$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

Didived powers at Borel–Moore homology

$$\mathscr{F}_\alpha^{(k)} := q^{-d_\alpha \frac{k(k-1)}{4}}$$



$$\in \mathscr{H}_{k\alpha}^{\mathrm{BM}}$$

Proposition

In $\mathscr{H}_{c_\alpha^k}^{\mathrm{BM}}$ *the divided powers satisfy:*

$$\left(\mathscr{F}_\alpha^{(1)}\right)^k = [k]_{q_\alpha^{\frac{1}{2}}}! \mathscr{F}_\alpha^{(k)}$$

where $q_\alpha = q^{\frac{(\alpha,\alpha)}{2}}$, $[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}$, *and* $[x]_q! = [x]_q \cdots [1]_q$.

Quantum groups

$$\Pi = \{\alpha_1, \ldots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi\rangle$$

Homology

$$D =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \ldots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \ldots, z_{c(\alpha_i)}^{\alpha_i}\}$$

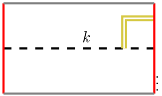
$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

Didived powers at Borel–Moore homology

$$\mathscr{F}_\alpha^{(k)} := q^{-d_\alpha \frac{k(k-1)}{4}}$$



$$\in \mathscr{H}^{\mathrm{BM}}_{k\alpha}$$

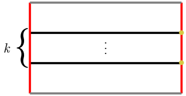
Proposition

In $\mathscr{H}^{\mathrm{BM}}_{c_\alpha^k}$ the divided powers satisfy:

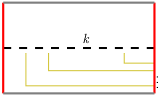
$$\left(\mathscr{F}_\alpha^{(1)}\right)^k = [k]_{q_\alpha^{\frac{1}{2}}}!\mathscr{F}_\alpha^{(k)}$$

where $q_\alpha = q^{\frac{(\alpha,\alpha)}{2}}$, $[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}$, *and* $[x]_q! = [x]_q \cdots [1]_q$.

Idea: In Borel–Moore homology one has:



$$= (k)_{q^{d_\alpha}}! \tag{3}$$



,

where $(x)_q = 1 + q + \cdots + q^{x-1}$ and $(x)_q! = (x)_q(x-1)_q \cdots (1)_q$.

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D \quad =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$
$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

A free algebra in standard homologies

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D \quad =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

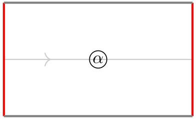
$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

A free algebra in standard homologies

$$\mathscr{F}_\alpha :=$$



$$\in \mathscr{H}_\alpha.$$

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$D \quad =$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

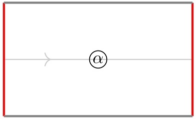
$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

A free algebra in standard homologies

$\mathscr{F}_\alpha :=$



$\in \mathscr{H}_\alpha.$

Theorem (Bigelow–M.)

The algebra $\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$ *is the free* $\mathscr{R}$*-algebra generated by* $\{\mathscr{F}_\alpha, \alpha \in \Pi\}$.

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D \quad =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

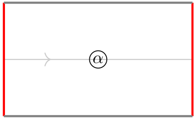
$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

A free algebra in standard homologies

$$\mathscr{F}_\alpha :=$$



$$\in \mathscr{H}_\alpha.$$

Theorem (Bigelow–M.)

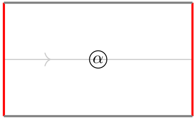
The algebra $\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$ *is the free* $\mathscr{R}$*-algebra generated by* $\{\mathscr{F}_\alpha, \alpha \in \Pi\}$.

Idea:

- Poincaré duality provides a pairing $\mathscr{H}_c \otimes \mathscr{H}_c^{\mathrm{BM}} \to \mathscr{R}$,

A free algebra in standard homologies

$\mathscr{F}_\alpha :=$



$\in \mathscr{H}_\alpha .$

Theorem (Bigelow–M.)

The algebra $\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$ *is the free* $\mathscr{R}$*-algebra generated by* $\{\mathscr{F}_\alpha, \alpha \in \Pi\}$.

Idea:

- Poincaré duality provides a pairing $\mathscr{H}_c \otimes \mathscr{H}_c^{\mathrm{BM}} \to \mathscr{R}$,
- (Godfard) $\mathscr{H}_c$ is a free $\mathscr{R}$-module,

Quantum groups

$$\Pi = \{\alpha_1, \ldots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$D \;=$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \ldots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \ldots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi\rangle$$

Homology

$D =$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

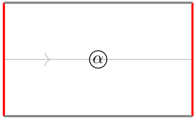
$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

A free algebra in standard homologies

$\mathscr{F}_\alpha :=$



$\in \mathscr{H}_\alpha.$

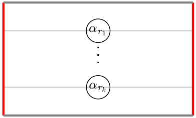
Theorem (Bigelow–M.)

The algebra $\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$ *is the free* $\mathscr{R}$*-algebra generated by* $\{\mathscr{F}_\alpha, \alpha \in \Pi\}$.

Idea:

- Poincaré duality provides a pairing $\mathscr{H}_c \otimes \mathscr{H}_c^{\mathrm{BM}} \to \mathscr{R}$,
- (Godfard) $\mathscr{H}_c$ is a free $\mathscr{R}$-module,
- The family

$\mathscr{F}_{\alpha_{r_1}} \cdots \mathscr{F}_{\alpha_{r_k}} =$



.

is dual to $\{\mathscr{F}_{(r_1,\dots,r_m)}$ s.t. $\sum \alpha_{r_i} = c\}$.

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D \quad =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

Quantum Serre relation at Borel–Moore homology

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi\rangle$$

Homology

$$D \quad =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

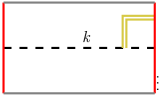
$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

Quantum Serre relation at Borel–Moore homology

Recall:

$$\mathscr{F}_\alpha^{(k)} := q^{-d_\alpha \frac{k(k-1)}{4}}$$



$$\in \mathscr{H}_{k\alpha}^{\mathrm{BM}}$$

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

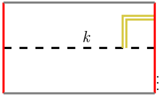
$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

Quantum Serre relation at Borel–Moore homology

Recall:

$$\mathscr{F}_\alpha^{(k)} := q^{-d_\alpha \frac{k(k-1)}{4}}$$



$$\in \mathscr{H}_{k\alpha}^{\mathrm{BM}}$$

Theorem (Homological and integral version of $U_q^{<0}\mathfrak{g}$)

- *One has:*

$$\sum_{l=0}^{1-a_{i,j}} (-1)^l \mathscr{F}_\alpha^{(l)} \mathscr{F}_\beta^{(1)} \mathscr{F}_\alpha^{((1-a_{i,j})-l)} = 0$$

Quantum groups

$$\Pi = \{\alpha_1, \ldots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D \quad =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \ldots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \ldots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[\mathsf{q}^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

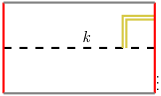
$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

Quantum Serre relation at Borel–Moore homology

Recall:

$$\mathscr{F}_\alpha^{(k)} := q^{-d_\alpha \frac{k(k-1)}{4}}$$



$$\in \mathscr{H}^{\mathrm{BM}}_{k\alpha}$$

Theorem (Homological and integral version of $U_q^{<0}\mathfrak{g}$)

- *One has:*

$$\sum_{l=0}^{1-a_{i,j}} (-1)^l \mathscr{F}_\alpha^{(l)} \mathscr{F}_\beta^{(1)} \mathscr{F}_\alpha^{((1-a_{i,j})-l)} = 0$$

- *Thus, if* $\mathsf{q} := q^{\frac{1}{2}}$, *there is an algebra morphism:*

$$\begin{array}{ccl} U_{\mathsf{q}}^{<0}\mathfrak{g} & \to & \mathscr{H}^{\mathrm{BM}} := \bigoplus_{c \in \mathbb{N}[\Pi]} \mathscr{H}_c^{\mathrm{BM}} \\ F_\alpha^{(k)} & \mapsto & \mathscr{F}_\alpha^{(k)} \text{ for all } \alpha \in \Pi \end{array} .$$

Above algebras are $\mathbb{Z}\left[\mathsf{q}^{\pm 1}\right]$ *resp.* $\mathbb{Z}\left[q^{\pm 1}\right]$ *algebras.*

Quantum groups

$$\Pi = \{\alpha_1, \ldots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D \quad =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \ldots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \ldots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

The quantum Serre class

$$D =$$

Let $S_c' \subset \mathrm{Conf}_c$ be the configurations with a point in either the red or the green.

Quantum groups

$$\Pi = \{\alpha_1, \ldots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$D \quad =$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \ldots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \ldots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

The quantum Serre class

$D =$

Let $S_c' \subset \mathrm{Conf}_c$ be the configurations with a point in either the red or the green.

Proposition

For any c we have:

$$\mathscr{H}_c^{\mathrm{BM}} \simeq \mathrm{H}_{|c|}^{\mathrm{BM}} \left(\mathrm{Conf}_c, S_c'; \mathscr{R}_c \right) := \mathscr{H}_c'$$

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D \quad =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

The quantum Serre class

$$D =$$

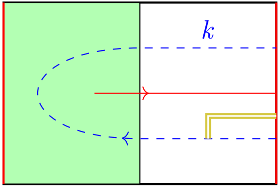
Let $S'_c \subset \mathrm{Conf}_c$ be the configurations with a point in either the red or the green.

Proposition

For any c we have:

$$\mathscr{H}_c^{\mathrm{BM}} \simeq \mathrm{H}_{|c|}^{\mathrm{BM}}\left(\mathrm{Conf}_c, S'_c; \mathscr{R}_c\right) := \mathscr{H}'_c$$

$$\mathrm{qSerre}_{\alpha,\beta}^{k} :=$$



$$\in \mathscr{H}'_{k\alpha+\beta}. \tag{4}$$

where we replace pearls by colors (and indices): blue for α, red for β.

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D \quad =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

Expressing qSerre in two ways

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D \;=$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

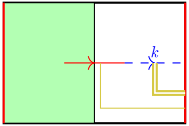
$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

Expressing qSerre in two ways

In $\mathscr{H}'_{k\alpha+\beta}$ we have:

$$\mathrm{qSerre}_{\alpha,\beta}^{(k)} = q_\alpha^{\frac{-k(k-1)}{2}} \prod_{l=0}^{k-1} \left(q_\alpha^{-l} q_{\alpha,\beta}^{-1} - q_{\alpha,\beta} \right)$$



Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi\rangle$$

Homology

$$D \quad =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$
$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

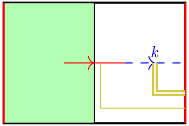
$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

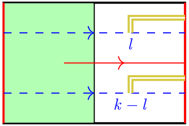
Expressing qSerre in two ways

In $\mathscr{H}'_{k\alpha+\beta}$ we have:

$$\mathrm{qSerre}^{(k)}_{\alpha,\beta} = q_\alpha^{\frac{-k(k-1)}{2}} \prod_{l=0}^{k-1} \left(q_\alpha^{-l} q_{\alpha,\beta}^{-1} - q_{\alpha,\beta} \right)$$



$$\mathrm{qSerre}^{(k)}_{\alpha,\beta} = \sum_{l=0}^{k} (-1)^{k-l} q_{\alpha,\beta}^{-l} q_\alpha^{-l(l-1)/2}$$



Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi\rangle$$

Homology

$$D \; =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

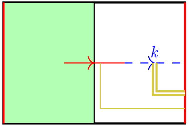
$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

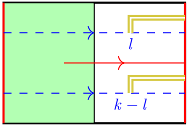
Expressing qSerre in two ways

In $\mathscr{H}'_{k\alpha+\beta}$ we have:

$$\mathrm{qSerre}^{(k)}_{\alpha,\beta} = q_\alpha^{\frac{-k(k-1)}{2}} \prod_{l=0}^{k-1} \left(q_\alpha^{-l} q_{\alpha,\beta}^{-1} - q_{\alpha,\beta} \right)$$

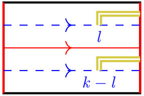


$$\mathrm{qSerre}^{(k)}_{\alpha,\beta} = \sum_{l=0}^{k} (-1)^{k-l} q_{\alpha,\beta}^{-l} q_\alpha^{-l(l-1)/2}$$

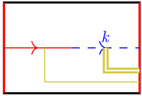


Finally, returning right terms in $\mathscr{H}'_{k\alpha+\beta}$:

$$\sum_{l=0}^{k} (-1)^{k-l} q_{\alpha,\beta}^{-l} q_\alpha^{-l(l-1)/2}$$



$$= q_\alpha^{\frac{k(1-k)}{2}} \prod_{l=0}^{k-1} \left(q_\alpha^{-l} q_{\alpha,\beta}^{-1} - q_{\alpha,\beta} \right)$$



Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

D =

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$
$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

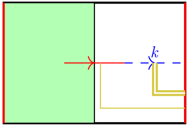
$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

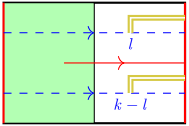
Expressing qSerre in two ways

In $\mathscr{H}'_{k\alpha+\beta}$ we have:

$$\mathrm{qSerre}_{\alpha,\beta}^{(k)} = q_\alpha^{\frac{-k(k-1)}{2}} \prod_{l=0}^{k-1} \left(q_\alpha^{-l} q_{\alpha,\beta}^{-1} - q_{\alpha,\beta} \right)$$

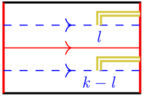


$$\mathrm{qSerre}_{\alpha,\beta}^{(k)} = \sum_{l=0}^{k} (-1)^{k-l} q_{\alpha,\beta}^{-l} q_\alpha^{-l(l-1)/2}$$

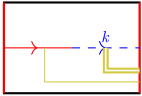


Finally, returning right terms in $\mathscr{H}'_{k\alpha+\beta}$:

$$\sum_{l=0}^{k} (-1)^{k-l} q_{\alpha,\beta}^{-l} q_\alpha^{-l(l-1)/2}$$



$$= q_\alpha^{\frac{k(1-k)}{2}} \prod_{l=0}^{k-1} \left(q_\alpha^{-l} q_{\alpha,\beta}^{-1} - q_{\alpha,\beta} \right)$$



If $k = 1 - a_{i,j}$, the RHS is 0, the LHS is $\sum_{l=0}^{1-a_{i,j}} (-1)^l \mathscr{F}_\alpha^{(l)} \mathscr{F}_\beta^{(1)} \mathscr{F}_\alpha^{((1-a_{i,j})-l)}$ □

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D \quad =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

In between homologies

There is a canonical map:

$$\iota_c : \mathscr{H}_c \to \mathscr{H}_c^{\mathrm{BM}},$$

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

In between homologies

There is a canonical map:

$$\iota_c : \mathscr{H}_c \to \mathscr{H}_c^{\mathrm{BM}},$$

We fix: $\overline{\mathscr{H}}_c := \mathrm{Im}(\iota_c) \subset \mathscr{H}_c^{\mathrm{BM}}$, and $\overline{\mathscr{H}} := \bigoplus_{c \in \mathbb{N}\Pi} \overline{\mathscr{H}}_c$.

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

In between homologies

There is a canonical map:

$$\iota_c : \mathscr{H}_c \to \mathscr{H}_c^{\mathrm{BM}},$$

We fix: $\overline{\mathscr{H}}_c := \mathrm{Im}(\iota_c) \subset \mathscr{H}_c^{\mathrm{BM}}$, and $\overline{\mathscr{H}} := \bigoplus_{c \in \mathbb{N}\Pi} \overline{\mathscr{H}}_c$.

Theorem (Bigelow–M.)

The space $\overline{\mathscr{H}}$ *is a* $\mathbb{Q}(q)$*-subalgebra of* $\mathscr{H}^{\mathrm{BM}}$ *which is isomorphic to* $U_q\mathfrak{g}^{<0}$*.*

In between homologies

There is a canonical map:

$$\iota_c : \mathscr{H}_c \to \mathscr{H}_c^{\mathrm{BM}},$$

We fix: $\overline{\mathscr{H}}_c := \mathrm{Im}(\iota_c) \subset \mathscr{H}_c^{\mathrm{BM}}$, and $\overline{\mathscr{H}} := \bigoplus_{c\in\mathbb{N}\Pi} \overline{\mathscr{H}}_c$.

Theorem (Bigelow–M.)

The space $\overline{\mathscr{H}}$ is a $\mathbb{Q}(q)$-subalgebra of $\mathscr{H}^{\mathrm{BM}}$ which is isomorphic to $U_q\mathfrak{g}^{<0}$.

Idea: $U_q\mathfrak{g}^{<0}$ is the quotient of the free algebra isomorphic to $\mathscr{H}$ by the radical of a particular pairing (Lusztig)

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c\in\mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D \quad = \boxed{}$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$
$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

In between homologies

There is a canonical map:

$$\iota_c : \mathscr{H}_c \to \mathscr{H}_c^{\mathrm{BM}},$$

We fix: $\overline{\mathscr{H}}_c := \mathrm{Im}(\iota_c) \subset \mathscr{H}_c^{\mathrm{BM}}$, and $\overline{\mathscr{H}} := \bigoplus_{c \in \mathbb{N}\Pi} \overline{\mathscr{H}}_c$.

Theorem (Bigelow–M.)

The space $\overline{\mathscr{H}}$ *is a* $\mathbb{Q}(q)$*-subalgebra of* $\mathscr{H}^{\mathrm{BM}}$ *which is isomorphic to* $U_q\mathfrak{g}^{<0}$*.*

Idea: $U_q\mathfrak{g}^{<0}$ is the quotient of the free algebra isomorphic to $\mathscr{H}$ by the radical of a particular pairing (Lusztig)

Poincaré duality gives:

$$\mathscr{H}_c \otimes \mathscr{H}_c^{\mathrm{BM}} \to \mathscr{R},$$

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D \quad =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$
$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

In between homologies

There is a canonical map:

$$\iota_c : \mathscr{H}_c \to \mathscr{H}_c^{\mathrm{BM}},$$

We fix: $\overline{\mathscr{H}}_c := \mathrm{Im}(\iota_c) \subset \mathscr{H}_c^{\mathrm{BM}}$, and $\overline{\mathscr{H}} := \bigoplus_{c \in \mathbb{N}\Pi} \overline{\mathscr{H}}_c$.

Theorem (Bigelow–M.)

The space $\overline{\mathscr{H}}$ is a $\mathbb{Q}(q)$-subalgebra of $\mathscr{H}^{\mathrm{BM}}$ which is isomorphic to $U_q\mathfrak{g}^{<0}$.

Idea: $U_q\mathfrak{g}^{<0}$ is the quotient of the free algebra isomorphic to $\mathscr{H}$ by the radical of a particular pairing (Lusztig)
Poincaré duality gives:

$$\mathscr{H}_c \otimes \mathscr{H}_c^{\mathrm{BM}} \to \mathscr{R},$$

which precomposed by $1 \otimes \iota_c$ gives a (no longer perfect) pairing:

$$(\cdot, \cdot) : \mathscr{H}_c \otimes \mathscr{H}_c \to \mathscr{R},$$

and $\overline{\mathscr{H}}_c = \mathrm{Im}(\iota_c)$ is the quotient of $\mathscr{H}_c$ by the left radical.

Quantum groups

$$\Pi = \{\alpha_1, \ldots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D \quad =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \ldots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \ldots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

In between homologies

There is a canonical map:

$$\iota_c : \mathscr{H}_c \to \mathscr{H}_c^{\mathrm{BM}},$$

We fix: $\overline{\mathscr{H}}_c := \mathrm{Im}(\iota_c) \subset \mathscr{H}_c^{\mathrm{BM}}$, and $\overline{\mathscr{H}} := \bigoplus_{c \in \mathbb{N}\Pi} \overline{\mathscr{H}}_c$.

Theorem (Bigelow–M.)

The space $\overline{\mathscr{H}}$ is a $\mathbb{Q}(q)$-subalgebra of $\mathscr{H}^{\mathrm{BM}}$ which is isomorphic to $U_q\mathfrak{g}^{<0}$.

Idea: $U_q\mathfrak{g}^{<0}$ is the quotient of the free algebra isomorphic to $\mathscr{H}$ by the radical of a particular pairing (Lusztig)
Poincaré duality gives:

$$\mathscr{H}_c \otimes \mathscr{H}_c^{\mathrm{BM}} \to \mathscr{R},$$

which precomposed by $1 \otimes \iota_c$ gives a (no longer perfect) pairing:

$$(\cdot,\cdot) : \mathscr{H}_c \otimes \mathscr{H}_c \to \mathscr{R},$$

and $\overline{\mathscr{H}}_c = \mathrm{Im}(\iota_c)$ is the quotient of $\mathscr{H}_c$ by the left radical. Is this Lusztig pairing ?

A twisted coproduct

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D \quad =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$
$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

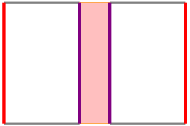
$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

A twisted coproduct

$D =$



Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$D =$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

Quantum groups

$$\Pi = \{\alpha_1, \ldots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$D =$ 

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \ldots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$
$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \ldots, z_{c(\alpha_i)}^{\alpha_i}\}$$

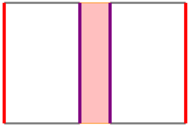
$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

A twisted coproduct

$D =$ 

Let $T_c \subset \mathrm{Conf}_c$ be the configurations with a point in the tricolor central band.

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$D =$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

A twisted coproduct

$D =$

Let $T_c \subset \mathrm{Conf}_c$ be the configurations with a point in the tricolor central band.

$$r_c : \mathscr{H}_c \to \mathrm{H}_{|c|}(\mathrm{Conf}_c, S_c \cup T_c).$$

Quantum groups

$$\Pi = \{\alpha_1, \ldots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi\rangle$$

Homology

$D \quad =$ 

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \ldots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$
$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \ldots, z_{c(\alpha_i)}^{\alpha_i}\}$$

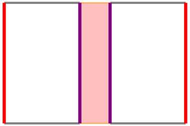
$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

A twisted coproduct

$D =$ 

Let $T_c \subset \mathrm{Conf}_c$ be the configurations with a point in the tricolor central band.

$$r_c : \mathscr{H}_c \to \mathrm{H}_{|c|}(\mathrm{Conf}_c, S_c \cup T_c).$$

Let $O_c \subset T_c$ be configurations with a point in the pink,

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$D =$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$
$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

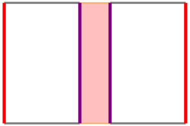
$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

A twisted coproduct

$D =$



Let $T_c \subset \mathrm{Conf}_c$ be the configurations with a point in the tricolor central band.

$$r_c : \mathscr{H}_c \to \mathrm{H}_{|c|}(\mathrm{Conf}_c, S_c \cup T_c).$$

Let $O_c \subset T_c$ be configurations with a point in the pink, it can be excised:

$$\mathrm{H}_{|c|}(\mathrm{Conf}_c, S_c \cup T_c) \simeq \mathrm{H}_{|c|}(\mathrm{Conf}_c \setminus O_c, (S_c \cup T_c) \setminus O_c).$$

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D \quad =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

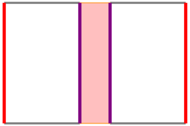
$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

A twisted coproduct

$$D =$$



Let $T_c \subset \mathrm{Conf}_c$ be the configurations with a point in the tricolor central band.

$$r_c : \mathscr{H}_c \to \mathrm{H}_{|c|}(\mathrm{Conf}_c, S_c \cup T_c).$$

Let $O_c \subset T_c$ be configurations with a point in the pink, it can be excised:

$$\mathrm{H}_{|c|}(\mathrm{Conf}_c, S_c \cup T_c) \simeq \mathrm{H}_{|c|}(\mathrm{Conf}_c \setminus O_c, (S_c \cup T_c) \setminus O_c).$$

The disk is split, namely:

$$(\mathrm{Conf}_c \setminus O_c, (S_c \cup T_c) \setminus O_c) = \bigsqcup_{c_1 + c_2 = c} \big(\mathrm{Conf}_{c_1} \times \mathrm{Conf}_{c_2}, S_c^L \times \mathrm{Conf}_{c_2} \cup \mathrm{Conf}_{c_1} \times S_c^R$$

where S_c^L is configurations with a point in one of both sides of the Left disk.

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$D =$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

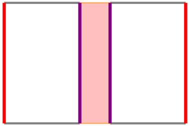
$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

A twisted coproduct

$D =$



Let $T_c \subset \mathrm{Conf}_c$ be the configurations with a point in the tricolor central band.

$$r_c : \mathscr{H}_c \to \mathrm{H}_{|c|}(\mathrm{Conf}_c, S_c \cup T_c).$$

Let $O_c \subset T_c$ be configurations with a point in the pink, it can be excised:

$$\mathrm{H}_{|c|}(\mathrm{Conf}_c, S_c \cup T_c) \simeq \mathrm{H}_{|c|}(\mathrm{Conf}_c \setminus O_c, (S_c \cup T_c) \setminus O_c).$$

The disk is split, namely:

$$(\mathrm{Conf}_c \setminus O_c, (S_c \cup T_c) \setminus O_c) = \bigsqcup_{c_1 + c_2 = c} \big(\mathrm{Conf}_{c_1} \times \mathrm{Conf}_{c_2}, S_c^L \times \mathrm{Conf}_{c_2} \cup \mathrm{Conf}_{c_1} \times S_c^R$$

where S_c^L is configurations with a point in one of both sides of the Left disk.
The relative Künneth formula gives:

$$r_c : \mathscr{H}_c \to \mathrm{H}_{|c|}(\mathrm{Conf}_c, S_c \cup T_c) \simeq \bigoplus_{c_1 + c_2 = c} \mathscr{H}_{c_1} \otimes \mathscr{H}_{c_2}$$

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi\rangle$$

Homology

$D =$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$
$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

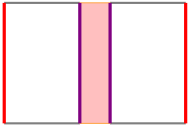
$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

A twisted coproduct

$D =$



Let $T_c \subset \mathrm{Conf}_c$ be the configurations with a point in the tricolor central band.

$$r_c : \mathscr{H}_c \to \mathrm{H}_{|c|}(\mathrm{Conf}_c, S_c \cup T_c).$$

Let $O_c \subset T_c$ be configurations with a point in the pink, it can be excised:

$$\mathrm{H}_{|c|}(\mathrm{Conf}_c, S_c \cup T_c) \simeq \mathrm{H}_{|c|}(\mathrm{Conf}_c \setminus O_c, (S_c \cup T_c) \setminus O_c).$$

The disk is split, namely:

$$(\mathrm{Conf}_c \setminus O_c, (S_c \cup T_c) \setminus O_c) = \bigsqcup_{c_1 + c_2 = c} \left(\mathrm{Conf}_{c_1} \times \mathrm{Conf}_{c_2}, S_c^L \times \mathrm{Conf}_{c_2} \cup \mathrm{Conf}_{c_1} \times S_c^R\right.$$

where S_c^L is configurations with a point in one of both sides of the Left disk.
The relative Künneth formula gives:

$$r_c : \mathscr{H}_c \to \mathrm{H}_{|c|}(\mathrm{Conf}_c, S_c \cup T_c) \simeq \bigoplus_{c_1 + c_2 = c} \mathscr{H}_{c_1} \otimes \mathscr{H}_{c_2}$$

Then $\bigoplus_c r_c : \mathscr{H} \to \mathscr{H} \otimes \mathscr{H}$ is Lusztig twisted coproduct on the free algebra $\mathscr{H}$.

Quantum groups

$$\Pi = \{\alpha_1, \ldots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \ldots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \ldots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

Compatibility between $(\cdot, \cdot)$ and r

$r := \bigoplus_c r_c : \mathscr{H} \to \mathscr{H} \otimes \mathscr{H}$ is Lusztig twisted coproduct on the free algebra $\mathscr{H}$.

Quantum groups

$$\Pi = \{\alpha_1, \ldots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D \;=$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \ldots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$
$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \ldots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

Compatibility between $(\cdot,\cdot)$ and r

$r := \bigoplus_c r_c : \mathscr{H} \to \mathscr{H} \otimes \mathscr{H}$ is Lusztig twisted coproduct on the free algebra $\mathscr{H}$.

Fact: The bilinear form $(\cdot,\cdot)$ is the unique one (Lusztig) such that $(1,1) = 1$, and:

(a) $(\mathscr{F}_\alpha, \mathscr{F}_\beta) = \delta_{\alpha,\beta}\frac{1}{(1-q_\alpha^{-1})}$ for all $\alpha, \beta \in \Pi$.

(b) $(x, y'y'') = (r(x), y' \otimes y'')$ for all $x, y, y'' \in \mathscr{H}$.

(c) $(xx', y'') = (x \otimes x', r(y''))$ for all $x, x', y'' \in \mathscr{H}$.

where the bilinear form on tensor products is given by:

$$(x_1 \otimes x_2, x_1' \otimes x_2') = (x_1, x_1')(x_2, x_2').$$

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

Compatibility between $(\cdot,\cdot)$ and r

$r := \bigoplus_c r_c : \mathscr{H} \to \mathscr{H} \otimes \mathscr{H}$ is Lusztig twisted coproduct on the free algebra $\mathscr{H}$.

Fact: The bilinear form $(\cdot,\cdot)$ is the unique one (Lusztig) such that $(1,1) = 1$, and:

(a) $(\mathscr{F}_\alpha, \mathscr{F}_\beta) = \delta_{\alpha,\beta} \frac{1}{(1-q_\alpha^{-1})}$ for all $\alpha, \beta \in \Pi$.

(b) $(x, y'y'') = (r(x), y' \otimes y'')$ for all $x, y, y'' \in \mathscr{H}$.

(c) $(xx', y'') = (x \otimes x', r(y''))$ for all $x, x', y'' \in \mathscr{H}$.

where the bilinear form on tensor products is given by:

$$(x_1 \otimes x_2, x_1' \otimes x_2') = (x_1, x_1')(x_2, x_2').$$

Finally, we have: $\overline{\mathscr{H}} = \mathscr{H} / \ker((\cdot, \mathscr{H})) = U_q\mathfrak{g}^{<0}$. $\square$

Quantum groups

$$\Pi = \{\alpha_1, \ldots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \ldots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \ldots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

Compatibility between $(\cdot,\cdot)$ and r

$r := \bigoplus_c r_c : \mathscr{H} \to \mathscr{H} \otimes \mathscr{H}$ is Lusztig twisted coproduct on the free algebra $\mathscr{H}$.

Fact: The bilinear form $(\cdot,\cdot)$ is the unique one (Lusztig) such that $(1,1) = 1$, and:

(a) $(\mathscr{F}_\alpha, \mathscr{F}_\beta) = \delta_{\alpha,\beta} \frac{1}{(1-q_\alpha^{-1})}$ for all $\alpha, \beta \in \Pi$.

(b) $(x, y'y'') = (r(x), y' \otimes y'')$ for all $x, y, y'' \in \mathscr{H}$.

(c) $(xx', y'') = (x \otimes x', r(y''))$ for all $x, x', y'' \in \mathscr{H}$.

where the bilinear form on tensor products is given by:

$$(x_1 \otimes x_2, x_1' \otimes x_2') = (x_1, x_1')(x_2, x_2').$$

Finally, we have: $\overline{\mathscr{H}} = \mathscr{H} / \ker((\cdot, \mathscr{H})) = U_q\mathfrak{g}^{<0}$. $\square$

In Progress:

- Reconstructing a full Borel (with K's), put a Hopf algebra structure.

Compatibility between $(\cdot,\cdot)$ and r

$r := \bigoplus_c r_c : \mathscr{H} \to \mathscr{H} \otimes \mathscr{H}$ is Lusztig twisted coproduct on the free algebra $\mathscr{H}$.

Fact: The bilinear form $(\cdot,\cdot)$ is the unique one (Lusztig) such that $(1,1)=1$, and:

(a) $(\mathscr{F}_\alpha, \mathscr{F}_\beta) = \delta_{\alpha,\beta} \frac{1}{(1-q_\alpha^{-1})}$ for all $\alpha, \beta \in \Pi$.

(b) $(x, y'y'') = (r(x), y' \otimes y'')$ for all $x, y, y'' \in \mathscr{H}$.

(c) $(xx', y'') = (x \otimes x', r(y''))$ for all $x, x', y'' \in \mathscr{H}$.

where the bilinear form on tensor products is given by:

$$(x_1 \otimes x_2, x_1' \otimes x_2') = (x_1, x_1')(x_2, x_2').$$

Finally, we have: $\overline{\mathscr{H}} = \mathscr{H} / \ker((\cdot, \mathscr{H})) = U_q\mathfrak{g}^{<0}$. $\square$

In Progress:

- Reconstructing a full Borel (with K's), put a Hopf algebra structure.
- All set for a Drinfel'd double construction of $U_q\mathfrak{g}$.

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D \; =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

Compatibility between $(\cdot,\cdot)$ and r

$r := \bigoplus_c r_c : \mathscr{H} \to \mathscr{H} \otimes \mathscr{H}$ is Lusztig twisted coproduct on the free algebra $\mathscr{H}$.

Fact: The bilinear form $(\cdot,\cdot)$ is the unique one (Lusztig) such that $(1,1) = 1$, and:

(a) $(\mathscr{F}_\alpha, \mathscr{F}_\beta) = \delta_{\alpha,\beta} \frac{1}{(1-q_\alpha^{-1})}$ for all $\alpha, \beta \in \Pi$.

(b) $(x, y'y'') = (r(x), y' \otimes y'')$ for all $x, y, y'' \in \mathscr{H}$.

(c) $(xx', y'') = (x \otimes x', r(y''))$ for all $x, x', y'' \in \mathscr{H}$.

where the bilinear form on tensor products is given by:

$$(x_1 \otimes x_2, x_1' \otimes x_2') = (x_1, x_1')(x_2, x_2').$$

Finally, we have: $\overline{\mathscr{H}} = \mathscr{H} / \ker((\cdot, \mathscr{H})) = U_q\mathfrak{g}^{<0}$. $\square$

In Progress:

- Reconstructing a full Borel (with K's), put a Hopf algebra structure.
- All set for a Drinfel'd double construction of $U_q\mathfrak{g}$.
- Relate with *stated skein* algebras on surfaces with marked boundary.

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi\rangle$$

Homology

$$D \quad =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$
$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

Recovering a PBW basis ?

How to find a basis of $\overline{\mathscr{H}}$?

Recovering a PBW basis ?

How to find a basis of $\overline{\mathscr{H}}$? An algebraic answer: Poincaré–Birkhoff–Witt basis.

Quantum groups

$$\Pi = \{\alpha_1, \ldots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D \; =$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \ldots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \ldots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$D =$

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

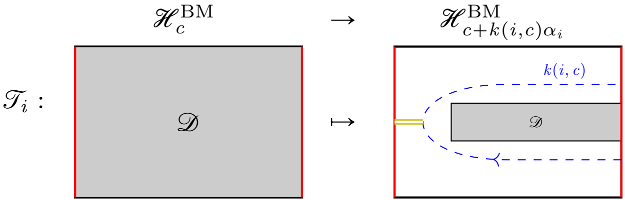
$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

Recovering a PBW basis ?

How to find a basis of $\overline{\mathscr{H}}$? An algebraic answer: Poincaré–Birkhoff–Witt basis.



where $k(i,c) := \sum_{\alpha_j \in \Pi} (-a_{i,j})$.

Proposition

For a given simple root $\alpha_i \in \Pi$, the map $\mathscr{T}_i$ restricted to

$$\bigoplus_{c, c(\alpha_i)=0} \overline{\mathscr{H}}_c$$

is the appropriate restriction of the algebra morphism associated with the corresponding braid in $\mathscr{B}_\mathfrak{g}$.

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D \;=$$

$$\mathrm{Conf}_c = \{(\boldsymbol{z^{\alpha_1}}, \dots, \boldsymbol{z^{\alpha_l}})\}$$

$$\boldsymbol{z^{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

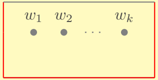
Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi\rangle$$

Homology

$D_n =$



$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$
$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

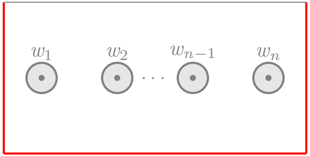
$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

Punctured disks



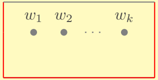
D_n is the n times punctured disk, D_n° is the disk with n holes.

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$D_n =$ 

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

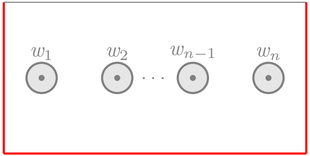
$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

Punctured disks



D_n is the n times punctured disk, D_n° is the disk with n holes. For $c \in \mathbb{N}\Pi$, the c-colored configuration space, defined as follows:

$$\mathrm{Conf}_c(D_n) := \left(D_n^{|c|} \setminus \bigcup_{i<j} \{z_i = z_j\} \right) \Big/ \mathfrak{S}_{c(\alpha_1)} \times \cdots \times \mathfrak{S}_{c(\alpha_l)}$$

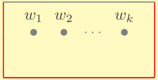
Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D_n =$$



$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$
$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

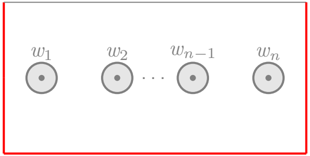
$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

Punctured disks



D_n is the n times punctured disk, D_n° is the disk with n holes. For $c \in \mathbb{N}\Pi$, the c-colored configuration space, defined as follows:

$$\mathrm{Conf}_c(D_n) := \left(D_n^{|c|} \setminus \bigcup_{i<j} \{z_i = z_j\} \right) \Big/ \mathfrak{S}_{c(\alpha_1)} \times \dots \times \mathfrak{S}_{c(\alpha_l)}$$

$S_c \subset \mathrm{Conf}_c(D_n)$ is now the set of configurations with a point in the (connected) red zone.

More winding numbers

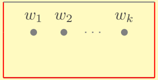
Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D_n =$$



$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

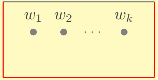
Quantum groups

$$\Pi = \{\alpha_1, \ldots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D_n =$$



$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \ldots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \ldots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

More winding numbers

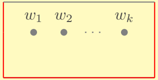
$$\Phi_c(D_n) : \begin{array}{rcl} \mathrm{Conf}_c & \to & (\mathbb{S}^1)^l \times (\mathbb{S}^1)^{\frac{l(l-1)}{2}} \times \left((\mathbb{S}^1)^l\right)^n \\ (\boldsymbol{z}^{\alpha_1}, \ldots, \boldsymbol{z}^{\alpha_l}) & \mapsto & \Phi_c \times \prod_{i=1}^n \left(g(\boldsymbol{z}^{\alpha_1}, w_i), \ldots, g(\boldsymbol{z}^{\alpha_l}, w_i)\right) \end{array}$$

Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$D_n =$ 

$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

More winding numbers

$$\Phi_c(D_n) : \begin{array}{ccc} \mathrm{Conf}_c & \to & (\mathbb{S}^1)^l \times (\mathbb{S}^1)^{\frac{l(l-1)}{2}} \times \left((\mathbb{S}^1)^l\right)^n \\ (\boldsymbol{z}^{\alpha_1}, \dots, \boldsymbol{z}^{\alpha_l}) & \mapsto & \Phi_c \times \prod_{i=1}^n \left(g(\boldsymbol{z}^{\alpha_1}, w_i), \dots, g(\boldsymbol{z}^{\alpha_l}, w_i)\right) \end{array}$$

$$\pi_c(D_n) := \pi_1\left((\mathbb{S}^1)^l \times (\mathbb{S}^1)^{\frac{l(l-1)}{2}} \times \left((\mathbb{S}^1)^l\right)^n\right) = \pi_c \times \prod_{1 \leqslant i \leqslant l, 1 \leqslant j \leqslant n} \mathbb{Z}\langle \mathfrak{w}(i,j) \rangle$$

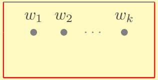
Quantum groups

$$\Pi = \{\alpha_1, \ldots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$D_n =$



$$\mathrm{Conf}_c = \{(\boldsymbol{z^{\alpha_1}}, \ldots, \boldsymbol{z^{\alpha_l}})\}$$

$$\boldsymbol{z^{\alpha_i}} = \{z_1^{\alpha_i}, \ldots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

More winding numbers

$$\Phi_c(D_n) : \begin{array}{rcl} \mathrm{Conf}_c & \to & (\mathbb{S}^1)^l \times (\mathbb{S}^1)^{\frac{l(l-1)}{2}} \times \left((\mathbb{S}^1)^l\right)^n \\ (\boldsymbol{z}^{\alpha_1}, \ldots, \boldsymbol{z}^{\alpha_l}) & \mapsto & \Phi_c \times \prod_{i=1}^n \left(g(\boldsymbol{z}^{\alpha_1}, w_i), \ldots, g(\boldsymbol{z}^{\alpha_l}, w_i)\right) \end{array}$$

$$\pi_c(D_n) := \pi_1\left((\mathbb{S}^1)^l \times (\mathbb{S}^1)^{\frac{l(l-1)}{2}} \times \left((\mathbb{S}^1)^l\right)^n\right) = \pi_c \times \prod_{1 \leqslant i \leqslant l, 1 \leqslant j \leqslant n} \mathbb{Z}\langle \mathfrak{w}(i,j) \rangle$$

Let $\mathscr{R}_{\boldsymbol{L}} := \mathbb{Z}\left[q^{\pm 1}, (s_j^i)^{\pm 1}\right]_{1 \leqslant i \leqslant n, 1 \leqslant j \leqslant l}$, and:

$$\varphi_c(D_n) : \begin{cases} \mathbb{Z}\left[\pi_c(D_n)\right] & \to & \mathscr{R}_{\boldsymbol{L}} \\ k_i & \mapsto & -q^{(\alpha_i, \alpha_i)/2} = -q^{d_i} \\ k_{(i,j)} & \mapsto & q^{(\alpha_i, \alpha_j)/2} \\ \mathfrak{w}(i,j) & \mapsto & s_{i,j}. \end{cases}$$

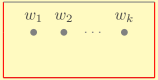
Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi\rangle$$

Homology

$D_n =$



$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}^{\mathrm{BM}}_{|c|}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

More winding numbers

$$\Phi_c(D_n) : \begin{array}{rcl} \mathrm{Conf}_c & \to & (\mathbb{S}^1)^l \times (\mathbb{S}^1)^{\frac{l(l-1)}{2}} \times \left((\mathbb{S}^1)^l\right)^n \\ (\boldsymbol{z}^{\alpha_1}, \dots, \boldsymbol{z}^{\alpha_l}) & \mapsto & \Phi_c \times \prod_{i=1}^n \left(g(\boldsymbol{z}^{\alpha_1}, w_i), \dots, g(\boldsymbol{z}^{\alpha_l}, w_i)\right) \end{array}$$

$$\pi_c(D_n) := \pi_1\left((\mathbb{S}^1)^l \times (\mathbb{S}^1)^{\frac{l(l-1)}{2}} \times \left((\mathbb{S}^1)^l\right)^n\right) = \pi_c \times \prod_{1 \leqslant i \leqslant l, 1 \leqslant j \leqslant n} \mathbb{Z}\langle \mathfrak{w}(i,j)\rangle$$

Let $\mathscr{R}_{\boldsymbol{L}} := \mathbb{Z}\left[q^{\pm 1}, (s_j^i)^{\pm 1}\right]_{1 \leqslant i \leqslant n, 1 \leqslant j \leqslant l}$, and:

$$\varphi_c(D_n) : \left\{ \begin{array}{rcl} \mathbb{Z}\left[\pi_c(D_n)\right] & \to & \mathscr{R}_{\boldsymbol{L}} \\ k_i & \mapsto & -q^{(\alpha_i, \alpha_i)/2} = -q^{d_i} \\ k_{(i,j)} & \mapsto & q^{(\alpha_i, \alpha_j)/2} \\ \mathfrak{w}(i,j) & \mapsto & s_{i,j}. \end{array} \right.$$

The data set $\mathscr{R}_c(D_n) := (\mathscr{R}_{\boldsymbol{L}}, \mathscr{W}_c(D_n), \varphi_c(D_n))$ provides a local system.

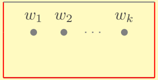
Quantum groups

$$\Pi = \{\alpha_1, \ldots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi\rangle$$

Homology

$$D_n =$$



$$\mathrm{Conf}_c = \{(\boldsymbol{z^{\alpha_1}}, \ldots, \boldsymbol{z^{\alpha_l}})\}$$

$$\boldsymbol{z^{\alpha_i}} = \{z_1^{\alpha_i}, \ldots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

More winding numbers

$$\Phi_c(D_n) : \begin{array}{rcl} \mathrm{Conf}_c & \to & (\mathbb{S}^1)^l \times (\mathbb{S}^1)^{\frac{l(l-1)}{2}} \times \left((\mathbb{S}^1)^l\right)^n \\ (\boldsymbol{z}^{\alpha_1}, \ldots, \boldsymbol{z}^{\alpha_l}) & \mapsto & \Phi_c \times \prod_{i=1}^n \left(g(\boldsymbol{z}^{\alpha_1}, w_i), \ldots, g(\boldsymbol{z}^{\alpha_l}, w_i)\right) \end{array}$$

$$\pi_c(D_n) := \pi_1\left((\mathbb{S}^1)^l \times (\mathbb{S}^1)^{\frac{l(l-1)}{2}} \times \left((\mathbb{S}^1)^l\right)^n\right) = \pi_c \times \prod_{1 \leqslant i \leqslant l, 1 \leqslant j \leqslant n} \mathbb{Z}\langle \mathfrak{w}(i,j)\rangle$$

Let $\mathscr{R}_{\boldsymbol{L}} := \mathbb{Z}\left[q^{\pm 1}, (s_j^i)^{\pm 1}\right]_{1 \leqslant i \leqslant n, 1 \leqslant j \leqslant l}$, and:

$$\varphi_c(D_n) : \begin{cases} \mathbb{Z}\left[\pi_c(D_n)\right] & \to & \mathscr{R}_{\boldsymbol{L}} \\ k_i & \mapsto & -q^{(\alpha_i,\alpha_i)/2} = -q^{d_i} \\ k_{(i,j)} & \mapsto & q^{(\alpha_i,\alpha_j)/2} \\ \mathfrak{w}(i,j) & \mapsto & s_{i,j}. \end{cases}$$

The data set $\mathscr{R}_c(D_n) := (\mathscr{R}_{\boldsymbol{L}}, \mathscr{W}_c(D_n), \varphi_c(D_n))$ provides a local system.

$$\overline{\mathscr{H}}(D_n) := \bigoplus_{c \in \mathbb{N}\Pi} \overline{\mathscr{H}}_c(D_n) = \bigoplus_{c \in \mathbb{N}\Pi} \mathrm{Im}\left(\mathrm{H}_{|c|}(\mathrm{Conf}_c, S_c; \mathscr{R}_c) \to \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c; \mathscr{R}_c)\right)$$

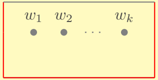
Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D_n =$$



$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

Recovering simple and Verma modules

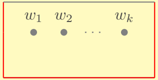
Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D_n =$$



$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

Recovering simple and Verma modules

Theorem (Bigelow–M.)

Spaces $\overline{\mathscr{H}}(D_n^\epsilon)$ *are* $U_q\mathfrak{g}$*-modules.*

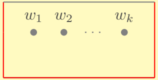
Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D_n =$$



$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

Recovering simple and Verma modules

Theorem (Bigelow–M.)

Spaces $\overline{\mathscr{H}}(D_n^\epsilon)$ are $U_q\mathfrak{g}$-modules. $\overline{\mathscr{H}}(D_1^\circ)$ and $\overline{\mathscr{H}}(D_1)$ are respectively the simple and the (co)Verma module of suitable highest weight.

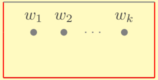
Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D_n =$$



$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

Recovering simple and Verma modules

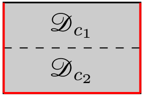
Theorem (Bigelow–M.)

Spaces $\overline{\mathscr{H}}(D_n^\epsilon)$ are $U_q\mathfrak{g}$-modules. $\overline{\mathscr{H}}(D_1^\circ)$ and $\overline{\mathscr{H}}(D_1)$ are respectively the simple and the (co)Verma module of suitable highest weight.

- $U_q\mathfrak{g}^{<0} = \overline{\mathscr{H}}$ acts by stacking disks from above:

$$\overline{\mathscr{H}} \times \overline{\mathscr{H}}(D_n) \to \overline{\mathscr{H}}(D_n)$$

$$(\mathscr{D}_{c_1}, \mathscr{D}_{c_2}) \mapsto$$



$$\in \overline{\mathscr{H}_{c_1+c_2}}(D_n).$$

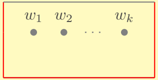
Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D_n =$$



$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

Recovering simple and Verma modules

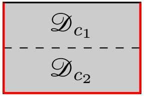
Theorem (Bigelow–M.)

Spaces $\overline{\mathscr{H}}(D_n^\epsilon)$ are $U_q\mathfrak{g}$-modules. $\overline{\mathscr{H}}(D_1^\circ)$ and $\overline{\mathscr{H}}(D_1)$ are respectively the simple and the (co)Verma module of suitable highest weight.

- $U_q\mathfrak{g}^{<0} = \overline{\mathscr{H}}$ acts by stacking disks from above:

$$\overline{\mathscr{H}} \times \overline{\mathscr{H}}(D_n) \to \overline{\mathscr{H}}(D_n)$$

$$(\mathscr{D}_{c_1}, \mathscr{D}_{c_2}) \mapsto$$



$$\in \overline{\mathscr{H}_{c_1+c_2}}(D_n).$$

- Let $S_c^2 \subset S_c \subset \mathrm{Conf}_c$ be configurations with two points in $\partial^- D_n$.

$$\mathscr{E}_\alpha : \mathscr{H}_c^{\mathrm{BM}} \xrightarrow{(-1)^{|c|}\partial_*} \mathrm{H}_{|c|-1}^{\mathrm{BM}}(S_c, S_c^2; \mathscr{R}_c) \simeq \bigoplus_{\alpha \in \Pi, c(\alpha) > 0} \mathscr{H}_{c-\alpha}^{\mathrm{BM}}(D_n) \to \mathscr{H}_{c-\alpha}^{\mathrm{BM}}(D_n)$$

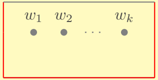
Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi\rangle$$

Homology

$D_n =$



$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

Recovering simple and Verma modules

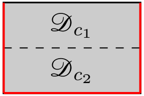
Theorem (Bigelow–M.)

Spaces $\overline{\mathscr{H}}(D_n^\epsilon)$ are $U_q\mathfrak{g}$-modules. $\overline{\mathscr{H}}(D_1^\circ)$ and $\overline{\mathscr{H}}(D_1)$ are respectively the simple and the (co)Verma module of suitable highest weight.

- $U_q\mathfrak{g}^{<0} = \overline{\mathscr{H}}$ acts by stacking disks from above:

$$\overline{\mathscr{H}} \times \overline{\mathscr{H}}(D_n) \to \overline{\mathscr{H}}(D_n)$$

$$(\mathscr{D}_{c_1}, \mathscr{D}_{c_2}) \mapsto$$



$$\in \overline{\mathscr{H}_{c_1+c_2}}(D_n).$$

- Let $S_c^2 \subset S_c \subset \mathrm{Conf}_c$ be configurations with two points in $\partial^- D_n$.

$$\mathscr{E}_\alpha : \mathscr{H}_c^{\mathrm{BM}} \xrightarrow{(-1)^{|c|}\partial_*} \mathrm{H}_{|c|-1}^{\mathrm{BM}}(S_c, S_c^2; \mathscr{R}_c) \simeq \bigoplus_{\alpha \in \Pi, c(\alpha) > 0} \mathscr{H}_{c-\alpha}^{\mathrm{BM}}(D_n) \to \mathscr{H}_{c-\alpha}^{\mathrm{BM}}(D_n)$$

- $\mathscr{K}_\alpha := \bigoplus_{c \in \mathbb{N}\Pi} \left(\prod_{p=1,\dots,n} s_{p,\alpha}^{-1}\right) \prod_{\beta \in \Pi} q_{\alpha,\beta}^{-c(\beta)} \, \mathrm{Id}_{\mathscr{H}_c^{\mathrm{BM}}(D_n)}$

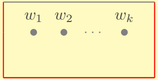
Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D_n =$$



$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$
$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

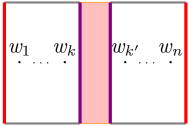
$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

Recovering tensor products

$$D_n =$$



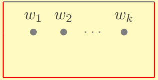
Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$D_n =$



$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

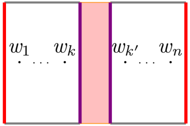
$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

Recovering tensor products

$D_n =$



Same kind of protocol as that for the twisted coproduct gives an isomorphism:

$$\mathscr{H}^{\mathrm{BM}}(D_n) \to \mathscr{H}^{\mathrm{BM}}(D_k) \otimes \mathscr{H}^{\mathrm{BM}}(D_{n-k}).$$

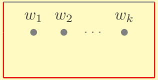
Quantum groups

$$\Pi = \{\alpha_1, \ldots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D_n =$$



$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \ldots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \ldots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

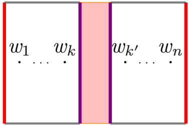
$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

Recovering tensor products

$$D_n =$$



Same kind of protocol as that for the twisted coproduct gives an isomorphism:

$$\mathscr{H}^{\mathrm{BM}}(D_n) \to \mathscr{H}^{\mathrm{BM}}(D_k) \otimes \mathscr{H}^{\mathrm{BM}}(D_{n-k}).$$

Theorem (Bigelow–M.)

The isomorphism is one of $U_q\mathfrak{g}$-modules while restricted to $\overline{\mathscr{H}}$. Namely: $\overline{\mathscr{H}}(D_n^\circ)$ is the product of n-copies of the simple module (with suitable highest weights).

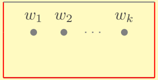
Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D_n =$$



$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$
$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

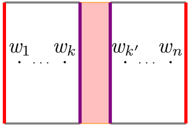
$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

Recovering tensor products

$$D_n =$$



Same kind of protocol as that for the twisted coproduct gives an isomorphism:

$$\mathscr{H}^{\mathrm{BM}}(D_n) \to \mathscr{H}^{\mathrm{BM}}(D_k) \otimes \mathscr{H}^{\mathrm{BM}}(D_{n-k}).$$

Theorem (Bigelow–M.)

The isomorphism is one of $U_q\mathfrak{g}$-modules while restricted to $\overline{\mathscr{H}}$. Namely: $\overline{\mathscr{H}}(D_n^\circ)$ is the product of n-copies of the simple module (with suitable highest weights).

Moreover, $\mathscr{H}^{\mathrm{BM}}(D_n)$ is naturally endowed with an action of the braid group on n strands $\mathscr{B}_n$ which is isomorphic to $\mathrm{Mod}(D_n)$. The action is shaped like an R-matrix.

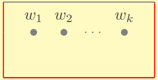
Quantum groups

$$\Pi = \{\alpha_1, \ldots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D_n =$$



$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \ldots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$
$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \ldots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

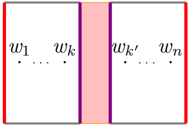
$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

Recovering tensor products

$$D_n =$$



Same kind of protocol as that for the twisted coproduct gives an isomorphism:

$$\mathscr{H}^{\mathrm{BM}}(D_n) \to \mathscr{H}^{\mathrm{BM}}(D_k) \otimes \mathscr{H}^{\mathrm{BM}}(D_{n-k}).$$

Theorem (Bigelow–M.)

The isomorphism is one of $U_q\mathfrak{g}$-modules while restricted to $\overline{\mathscr{H}}$. Namely: $\overline{\mathscr{H}}(D_n^\circ)$ is the product of n-copies of the simple module (with suitable highest weights).

Moreover, $\mathscr{H}^{\mathrm{BM}}(D_n)$ is naturally endowed with an action of the braid group on n strands $\mathscr{B}_n$ which is isomorphic to $\mathrm{Mod}(D_n)$. The action is shaped like an R-matrix.

In progress: It is given by the "universal" R-matrix.

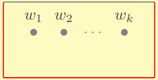
Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D_n =$$



$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

Merci pour votre attention !

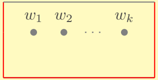
Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D_n =$$



$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$
$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

A dual action for E's

$$\begin{aligned} Y_c^i &:= \{\boldsymbol{z} \in \mathrm{Conf}_c(D_n), |\boldsymbol{z} \cap \partial^- D_n| \geqslant i\} \\ Y_c^i(\alpha) &:= \{\boldsymbol{z} \in \mathrm{Conf}_c(D_n), |\boldsymbol{z}^{\boldsymbol{\alpha}} \cap \partial^- D_n| \geqslant i\} \\ Y_c^+ = T(D_n) &:= \{\boldsymbol{z} \in \mathrm{Conf}_c(D_n), |\boldsymbol{z} \cap \partial^+ D_n| \geqslant 1\}. \end{aligned}$$

$$\mathrm{H}^*_{\mathrm{compact}}(\mathrm{Conf}_c(D_n), T(D_n); \mathscr{R}_c(D_n)) \to \mathrm{H}^*_{\mathrm{compact}}(Y_c^i(\alpha), Y_c^i(\alpha) \cap T(D_n)),$$

is induced by restriction. Notice that:

$$\partial Y_c^i(\alpha) = \left(Y_c^i(\alpha) \cap T(D_n)\right) \cup \left(Y_c^i(\alpha) \cap Y_c^{i+1}\right).$$

The relative Poincaré dual map gives ($\dim Y_c^i(\alpha) = 2m_c - i$):

$$\partial_c^i(\alpha) : \mathrm{H}_{m_c}(\mathrm{Conf}_c(D_n), S(D_n); \mathscr{R}_c(D_n)) \to \mathrm{H}_{m_c - i}(Y_c^i(\alpha), Y_c^i(\alpha) \cap Y_c^{i+1}; \mathscr{R}_c).$$

There is an isomorphism:

$$\mathrm{H}_{m_c - i}(Y_c^i(\alpha), Y_c^i(\alpha) \cap Y_c^{i+1}; \mathscr{R}_c) \simeq \mathscr{H}_{c - i\alpha}(D_n).$$

For $\alpha \in \Pi$, we define the k-th divided power of $\mathscr{E}_\alpha^{[1]}$ to be a map:

$$\mathscr{E}_\alpha^{[k]} : \mathscr{H}_c(D_n) \to \mathscr{H}_{c - k\alpha}(D_n)$$

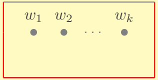
Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D_n =$$



$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

Left adjointness

Proposition

We recall the perfect pairing:

$$\langle \cdot, \cdot \rangle : \mathscr{H}(D_n) \times \mathscr{H}^{\mathrm{BM}}(D_n) \to \mathscr{R}(\boldsymbol{L}).$$

For this form, $\mathscr{K}_\alpha$ *is left adjoint to* $\mathscr{K}_\alpha^{-1}$, $\mathscr{E}_\alpha^{[k]}$ *is left adjoint to* $\mathscr{F}_\alpha^{(k)}$ *and* $\mathscr{F}_\alpha^{[1]}$ *is left adjoint to* $\mathscr{E}_\alpha$.

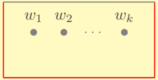
Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D_n =$$



$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

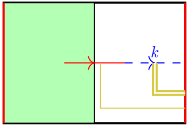
$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

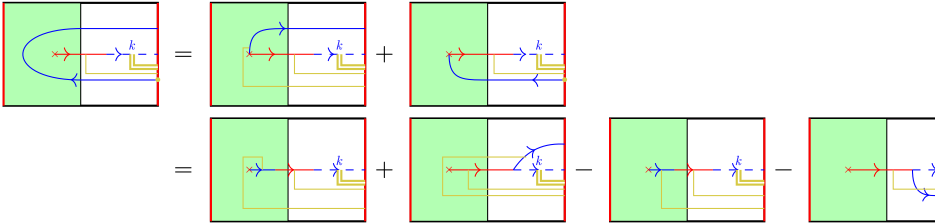
$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

Squashing $\mathrm{qSerre}_{\alpha,\beta}^{(k)}$

$$\mathrm{qSerre}_{\alpha,\beta}^{(k)} = q_\alpha^{\frac{-k(k-1)}{2}} \prod_{l=0}^{k-1} \left(q_\alpha^{-l} q_{\alpha,\beta}^{-1} - q_{\alpha,\beta} \right)$$



Idea:



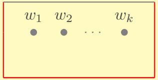
Quantum groups

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

$$U_q\mathfrak{g} = \langle E_\alpha, F_\alpha, K_\alpha, \alpha \in \Pi \rangle$$

Homology

$$D_n =$$



$$\mathrm{Conf}_c = \{(\boldsymbol{z}^{\boldsymbol{\alpha_1}}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha_l}})\}$$

$$\boldsymbol{z}^{\boldsymbol{\alpha_i}} = \{z_1^{\alpha_i}, \dots, z_{c(\alpha_i)}^{\alpha_i}\}$$

$$\mathscr{R}_c : \mathbb{Z}[\pi_1] \to \mathscr{R} = \mathbb{Z}[q^{\pm 1}]$$

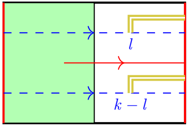
$$\mathscr{H}_c^{\mathrm{BM}} := \mathrm{H}_{|c|}^{\mathrm{BM}}(\mathrm{Conf}_c, S_c),$$

$$\mathscr{H} := \bigoplus_{c \in \mathbb{N}\Pi} \mathscr{H}_c$$

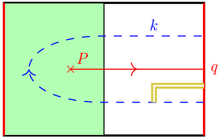
$$\mathscr{H} \otimes \mathscr{H}^{\mathrm{BM}} \to \mathscr{R}$$

Partitioning $\mathrm{qSerre}_{\alpha,\beta}^{(k)}$

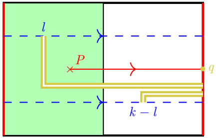
$$\mathrm{qSerre}_{\alpha,\beta}^{(k)} = \sum_{l=0}^{k} (-1)^{k-l} q_{\alpha,\beta}^{-l} q_\alpha^{-l(l-1)/2}$$



Idea: In $\mathscr{H}_c^{\mathrm{BM}'}$ we have:



$$= \sum_{l=0}^{k} (-1)^{k-l}$$



$$= \sum_{l=0}^{k} (-1)^{k-l} q_{\alpha,\beta}^{-l} q_\alpha^{-l(l-1)/2}$$